Lévy processes and applications - Infinite divisibility

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Characteristic function

Definition

The characteristic function of the random variable X (with values in $\mathbb{R}^d$), and distribution μ, is the function $\phi_X : \mathbb{R}^d \to \mathbb{C}$, defined by

$$\phi_X(u) = \mathbb{E}\left[e^{i(u \cdot X)}\right] = \int_{\mathbb{R}^d} e^{i(u \cdot x)} \mu(dx), \quad u \in \mathbb{R}^d.$$

- The characteristic function of a random variable completely characterizes its distribution, so we can write $\phi_X = \phi_\mu$.
- Properties of a characteristic function ϕ :
  1. $\phi(0) = 1$
  2. $|\phi(u)| \leq 1, \ \forall u \in \mathbb{R}^d$.
  3. ϕ is uniformly continuous
- The moments of a random variable are related to the derivatives at zero of its characteristic function - see Cont and Tankov, page 30.
- Exercise: Prove property 2 (use the fact that $\mathbb{E}\left[\left|e^{i(u \cdot X)}\right|\right] \leq \sqrt{\mathbb{E}\left[\left|e^{i(u \cdot X)}\right|^2\right]}$ (why?)).

Infinite divisibility

Definition

A probability distribution μ on $\mathbb{R}^d$ is said to be infinitely divisible if for any $n \in \mathbb{N}$, there exist n i.i.d. random variables $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ such that $Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ has distribution μ.

Definition

A r.v. X is infinitely divisible if its distribution μ is infinitely divisible. This means that

$$X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)},$$

where $Y_1^{(n)}, \ldots, Y_n^{(n)}$ are i.i.d., for each $n \in \mathbb{N}$.

Theorem

The distribution μ is infinitely divisible iff for all $n \in \mathbb{N}$, exists μ_n with charact. func. ϕ_n:

$$\phi_\mu (u) = (\phi_n (u))^n$$

for all $u \in \mathbb{R}^d$.

- idea of the proof: Let X be a r.v. with distribution μ and characteristic function ϕ_μ. Taking the i.i.d. $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ such that $X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$, by the independence of the $Y_i^{(n)}$,

$$\mathbb{E}\left[e^{iuX}\right] = \left(\mathbb{E}\left[e^{iuY_1^{(n)}}\right]\right)^n = (\phi_n (u))^n ,$$

  where $\phi_n (u)$ is the charact. function of $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$.
- Exercise: Let $\alpha > 0$, $\beta > 0$. Show that the gamma-(α, β) distribution

$$\mu_{\alpha,\beta} (dx) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx, \quad \text{with } x > 0,$$

  with characteristic function $\left(1 - \frac{iu}{\beta}\right)^{-\alpha}$, is an infinitely-divisible distribution.
- For a table with examples of characteristic functions, see Cont and Tankov, page 33.

Infinite divisibility - Examples

- In each example, we will find iid $Y_1^{(n)}, \ldots, Y_n^{(n)}$ such that $X \overset{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$.

Example

(Gaussian random variable) Let X be Gaussian random variable, with density:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}^d.$$

$X \sim N\left(m, \sigma^2\right)$.
One can show that

$$\phi_X(u) = \exp\left(imu - \frac{1}{2}\sigma^2 u^2\right).$$

Infinite divisibility - Examples

Example

(continued) Therefore:

$$(\phi_X(u))^{\frac{1}{n}} = \exp\left(i\frac{m}{n}u - \frac{1}{2}\frac{\sigma^2}{n}u^2\right).$$

and X is inf. divis. with $X \overset{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ and

$$Y_j^{(n)} \sim N\left(\frac{m}{n}, \frac{\sigma^2}{n}\right).$$

Infinite divisibility - Examples

Example

(Poisson r.v.) Let $d = 1$ and $X : \Omega \to \mathbb{N}_0$ with $X \sim Po(\lambda)$, i.e.

$$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

It is well known that $E[X] = Var[X] = \lambda$ and it is easy to verify that

$$\phi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right].$$

Therefore

$$(\phi_X(u))^{\frac{1}{n}} = \exp\left[\frac{\lambda}{n}\left(e^{iu} - 1\right)\right].$$

and X is inf. divis.with $X \overset{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ and

$$Y_j^{(n)} \sim Po\left(\frac{\lambda}{n}\right).$$

Example

(Compound Poisson r.v.) Let $\{Z(n), n \in \mathbb{N}\}$ be a sequence of iid r.v. with law μ_Z. Let $N \sim Po(\lambda)$ and independent of the $Z(n)'s$. Define

$$X = Z(1) + Z(2) + \cdots + Z(N) = \sum_{n=0}^{N} Z(n).$$

Let us prove that, for each $u \in \mathbb{R}^d$,

$$\phi_X(u) = \exp\left[\int_{\mathbb{R}^d} \left(e^{i(u,y)} - 1\right) \lambda \mu_Z(dy)\right]. \quad (1)$$

$$\phi_X(u) = E\left[e^{i(u,X)}\right] = \sum_{n=0}^{\infty} E\left[e^{i(u,Z(1)+Z(2)+\cdots+Z(N))} | N = n\right] P[N = n]$$

$$= \sum_{n=0}^{\infty} E\left[e^{i(u,Z(1)+Z(2)+\cdots+Z(n))}\right] \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda \phi_Z(u))^n}{n!}$$

$$= \exp\left[\lambda\left(\phi_Z(u) - 1\right)\right].$$

Infinite divisibility - Examples

Example

(Continued) Therefore, with $\phi_Z(u) = \int_{\mathbb{R}^d} e^{i(u,y)} \mu_Z(dy)$, we obtain (1).
We denote the Compound Poisson by $X \sim Po(\lambda, \mu_Z)$. We have

$$(\phi_X(u))^{\frac{1}{n}} = \exp\left[\frac{\lambda}{n}(\phi_Z(u) - 1)\right]$$

and X is inf. divis. with $X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ and

$$Y_j^{(n)} \sim Po\left(\frac{\lambda}{n}, \mu_Z\right).$$

- Exercise: Let $d = 1$. Show that if $X \sim Po(\lambda)$ then $\phi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right]$.

The Lévy measure

Definition

Let ν be a Borel measure defined on $\mathbb{R}^d - \{0\}$. We say that ν is a Lévy measure if

$$\int_{\mathbb{R}^d - \{0\}} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty \quad (2)$$

- Note that $\varepsilon^2 \leq |x|^2 \wedge 1$ when $0 < \varepsilon \leq 1$ and $|x| \geq \varepsilon$. Therefore, by (2), we have that
$$\nu\left[(-\varepsilon, \varepsilon)^c\right] < \infty, \qquad \text{for all } \varepsilon > 0.$$
- Note: Condition (2) is equivalent to
$$\int_{\mathbb{R}^d - \{0\}} \frac{|x|^2}{1 + |x|^2} \nu(dx) < \infty.$$
- Note: one can assume that $\nu(\{0\}) = 0$ and then ν is defined on $\mathbb{R}^d$.
- Exercise: Show that $\nu\left[(-\varepsilon, \varepsilon)^c\right] < \infty$, for all $\varepsilon > 0$.
- Exercise: Show that Condition (2) is equivalent to $\int_{\mathbb{R}^d - \{0\}} \frac{|x|^2}{1+|x|^2} \nu(dx) < \infty$.

Lévy-Khintchine formula

Theorem

(Lévy-Khintchine): A distribution μ *on* $\mathbb{R}^d$ *is infinitely divisible if exists a vector* $b \in \mathbb{R}^d$*, a* $d \times d$ *positive definite symmetric matrix A and a Lévy measure* ν *on* $\mathbb{R}^d - \{0\}$ *such that, for all* $u \in \mathbb{R}^d$*,*

$$\phi_\mu(u) = \exp\left\{ i(b,u) - \frac{1}{2}(u,Au) + \right.$$
$$\left. + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{|x|<1}(x) \right] \nu(dx) \right\}. \quad (3)$$

Conversely, any mapping of the form (3) is the characteristic function of an inf. divis. probability measure on $\mathbb{R}^d$*.*

Lévy-Khintchine formula

- (b, A, ν) are the characteristics of the inf. divis. distribution μ.
- $\eta := \log(\phi_\mu)$ is the Lévy symbol or characteristic exponent or Lévy exponent:

$$\eta(u) = i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{|x|<1}(x) \right] \nu(dx).$$

- We will not prove the first part of the theorem (difficult, but it can be proved as a by product of the Lévy-Itô decomposition - to be discussed later)
- We prove the second part.

Proof (2nd part)

- We need to prove that the r.h.s of (3) is a characteristic function.
- i) Let $\{U(n), n \in \mathbb{N}\} \subset \mathbb{R}^d$ be a sequence of Borel sets such that $U(n) \searrow 0$ and define

$$\phi_n(u) = \exp\left\{ i\left(b - \int_{U(n)^c \cap \{x:|x|<1\}} x\nu(dx), u\right) - \frac{1}{2}(u, Au) + \right.$$
$$\left. + \int_{U(n)^c} \left(e^{i(u,x)} - 1\right)\nu(dx)\right\}.$$

ii) Clearly, ϕ_n is the distribution of a sum of a Normal dist. with an independent compound Poisson dist. Therefore, it is infinit. divis.
iii) Clearly,

$$\phi_\mu(u) = \lim_{n\to\infty} \phi_n(u).$$

Proof (continued)

iv) In order to prove that ϕ_μ is a characteristic function, we aplly Lévy's continuity theorem (see below) and therefore we only need to prove that $\psi_\mu(u)$ is continuous at zero, with

$$\begin{aligned}\psi_\mu(u) &= \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{|x|<1}(x)\right]\nu(dx) \\ &= \int_{|x|<1} \left(e^{i(u,x)} - 1 - i(u,x)\right)\nu(dx) + \\ &+ \int_{|x|\geq 1} \left(e^{i(u,x)} - 1\right)\nu(dx).\end{aligned}$$

v) By Taylor's theorem, the Cauchy-Schwarz inequality and dominated convergence, we have:

$$\begin{aligned}|\psi_\mu(u)| &\leq \frac{1}{2}\int_{|x|<1} |(u,x)|^2 \nu(dx) + \int_{|x|\geq 1} \left|e^{i(u,x)} - 1\right|\nu(dx) \\ &\leq \frac{|u|^2}{2}\int_{|x|<1} |x|^2 \nu(dx) + \int_{|x|\geq 1} \left|e^{i(u,x)} - 1\right|\nu(dx) \to 0 \text{ as } u \to 0.\end{aligned}$$

vi) It is now easy to verify directly that μ is infin. divis. ■

Remarks

- This techique of taking the limits of sequences composed of sums of gaussians with independent compound Poissons is very important.
- The cut-off function $c(x) = x\mathbf{1}_{|x|<1}(x)$ in (3) could be replaced by other $c(x)$ such that $e^{i(u,x)} - 1 - i(u, c(x))$ is a ν-integrable function for each $u \in \mathbb{R}^d$. For instance, we could have $c(x) = \frac{x}{1+|x|^2}$.
- Gaussian case: $b = m$ (mean), $A =$covariance matrix, $\nu = 0$.
- Poisson case: $b = 0$, $A = 0$, $\nu = \lambda\delta_1$
- Compound Poisson case: $b = 0$, $A = 0$, $\nu = \lambda\mu$, $\lambda > 0$ and μ a probab. measure on $\mathbb{R}^d$

Remarks

- All infinitely divisible distributions can be constructed as weak limits of convolutions of Gaussians with independent Poisson processes (convolution of distributions corresponds to the sum of the r.v.). In fact, they can be obtained as weak limits of Compound Poissons only.
- More about these properties: Applebaum, pages 31-33.

Stable random variables

- The set of stable distributions is an important subclass of the set of inf. divis. distributions
- Let $d = 1$ and $\{Y_n, n \in \mathbb{N}\}$ be a sequence of iid r.v. We consider the general central limit problem. Define the rescaled partial sums sequence:

$$S_n = \frac{Y_1 + \cdots + Y_n - b_n}{\sigma_n}.$$

  where $\{b_n, n \in \mathbb{N}\}$: sequence of real numbers; $\{\sigma_n, n \in \mathbb{N}\}$: sequence of positive numbers.
- Problem: When exists a r.v. X such that

$$\lim_{n\to\infty} P(S_n \leq x) = \lim_{n\to\infty} P(X \leq x) \quad ? \tag{4}$$

  In that case S_n converges in distribution to X.
- Usual central limit theorem: if $b_n = nm$, $\sigma_n = \sqrt{n}\sigma$. Then $X \sim N(0,1)$.

Stable Random variables

- A r.v. is said to be stable if it arises as a limit as in (4)
- This is equivalent to:

Definition

A r.v. X is said to be stable if exist real valued sequences $\{c_n, n \in \mathbb{N}\}$, $\{d_n, n \in \mathbb{N}\}$ with each $c_n > 0$, such that

$$X_1 + \cdots + X_n \stackrel{d}{=} c_n X + d_n, \tag{5}$$

where $X_1, \ldots, X_n$ are independent copies of X. In particular, it is strictly stable if each $d_n = 0$.

- In fact, it can be proved that if X is stable then $\sigma_n = \sigma n^{\frac{1}{\alpha}}$ with $0 < \alpha \leq 2$. The parameter α is called the index of stability.
- (5) is equivalent to

$$\phi_X(u)^n = e^{iud_n}\phi_X(c_n u).$$

- All stable random variables are infinitely divisible (trivial consequence of (5)).

Stable Random variables

Theorem

If X is a stable r.v. then:

1. *when* $\alpha = 2$, $X \sim N(b, A)$
2. *when* $\alpha \neq 2$, $A = 0$ *and*

$$\nu(dx) = \begin{cases} \frac{c_1}{x^{1+\alpha}} dx & \text{if } x > 0 \\ \frac{c_2}{|x|^{1+\alpha}} dx & \text{if } x < 0. \end{cases}, \quad \text{where } c_1, c_2 \geq 0 \text{ and } c_1 + c_2 > 0.$$

Proof can be found in the Book of Sato, p. 80.

Stable random variables

Stable Random variables

Theorem

A r.v. X is stable if and only if exist $\sigma > 0$, $-1 \leq \beta \leq 1$ *and* $\mu \in \mathbb{R}$ *such that*

1. *when* $\alpha = 2$,

$$\phi_X(u) = \exp\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right);$$

2. *when* $\alpha \neq 1, 2$

$$\phi_X(u) = \exp\left(i\mu u - \sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right]\right)$$

3. *when* $\alpha = 1$,

$$\phi_X(u) = \exp\left(i\mu u - \sigma |u| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right]\right)$$

Proof can be found in Sato, p. 86.

Stable Random variables

- $E\left[X^2\right] < \infty$ if and only if $\alpha = 2$ (only if X is Gaussian).
- $E\left[|X|\right] < \infty$ if and only if $1 < \alpha \leq 2$.
- All stable r.v. X have densities f_X. In general, can be expressed in series form, but in 3 cases, we have a closed form.
- **Normal distribution:** $\alpha = 2$ and $X \sim N\left(\mu, \sigma^2\right)$.
- **Cauchy distribution:** $\alpha = 1$, $\beta = 0$, $f_X(x) = \frac{\sigma}{\pi\left[(x-\mu)^2+\sigma^2\right]}$.
- **Lévy distribution**: $\alpha = \frac{1}{2}$, $\beta = 1$,

$$f_X(x) = \left(\frac{\sigma}{2\pi}\right)^{\frac{1}{2}} \frac{1}{(x-\mu)^{\frac{3}{2}}} \exp\left[\frac{\sigma}{-2(x-\mu)}\right] \quad \text{for } x > \mu.$$

Stable Random variables

- Exercise: Let X and Y be independent standard normal random variables (with mean 0). Show that Z has a Cauchy distribution, where $Z = X/Y$ if $Y \neq 0$ and $Z = 0$ if $Y = 0$.
- Remark: if X is stable and symmetric then

$$\phi_X(u) = \exp\left(-\rho^\alpha |u|^\alpha\right) \quad \text{for all } 0 < \alpha \leq 2.$$

where $\rho = \sigma$ for $0 < \alpha < 2$ and $\rho = \frac{\sigma}{\sqrt{2}}$ when $\alpha = 2$.
- Important feature of stable laws:when $\alpha \neq 2$ the decay of the tails is polynomial (slow decay $\Longrightarrow$ "heavy tails") -(if $\alpha = 2$ the decay is exponential):

$$P[X > x] \sim \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}x} \text{ as } x \to \infty \text{ if } \alpha = 2,$$

$$\lim_{x\to+\infty} x^\alpha P[X > x] \sim C_\alpha \frac{1+\beta}{2} \sigma^\alpha \text{ if } \alpha \neq 2, \text{ with } C_\alpha > 1,$$

$$\lim_{x\to-\infty} x^\alpha P[X < -x] \sim C_\alpha \frac{1-\beta}{2} \sigma^\alpha \text{ if } \alpha \neq 2, \text{ with } C_\alpha > 1.$$

Stable Random variables

- All the previous results can be extended to random variables with values in $\mathbb{R}^d$. Just replace $X_1, \ldots, X_n, X$ and each d_n in (5) by vectors and adapt the previous theorems.
- Note that when $\alpha \neq 2$ and $d > 1$, then the Lévy measure is given by

$$\nu(dx) = \frac{c}{|x|^{d+\alpha}} dx, \quad \text{where } c > 0.$$

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