Models in Finance - Class 10

Master in Actuarial Science

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ISEG

Introduction to the valuation of derivatives

- Derivative: is a security or contract which promises to make a payment at a specified time in the future, the amount of which depends upon the behaviour of some underlying asset (or market variable) up to and including the time of the payment.
- Examples:
  1. Options on stocks: (underlying: price of stock shares).
  2. Futures contract on wheat (underlying: price of wheat).
  3. Forward contract for selling US dollars (underlying: exchange rate EUR-USD).
- Derivatives may depend on many other variables: interest rates, indexes, electricity price, etc...
- Other derivatives: Credit derivatives, insurance derivatives, weather derivatives, etc...

Arbitrage

- Arbitrage: an arbitrage opportunity is a strategy where we can make a sure profit with no risk ("a free lunch").
- Arbitrage strategy:
- (1) Start at time 0 with a portfolio which has a net value of zero (implying that we are long in some assets and short in others).
- (2) At some future time T: Probability (loss) $= 0$ and Probability (strictly positive profit) > 0.
- If an arbitrage opportunity exists, all the market participants would exploit it and the market prices of the assets in the portfolio would quickly change to remove the arbitrage opportunity.

Arbitrage

- Principle of No Arbitrage: arbitrage opportunities do not exist.
- If there are no arbitrage opportunities $\Longrightarrow$ any two securities or portfolios that give exactly the same payments must have the same price ("Law of One Price").

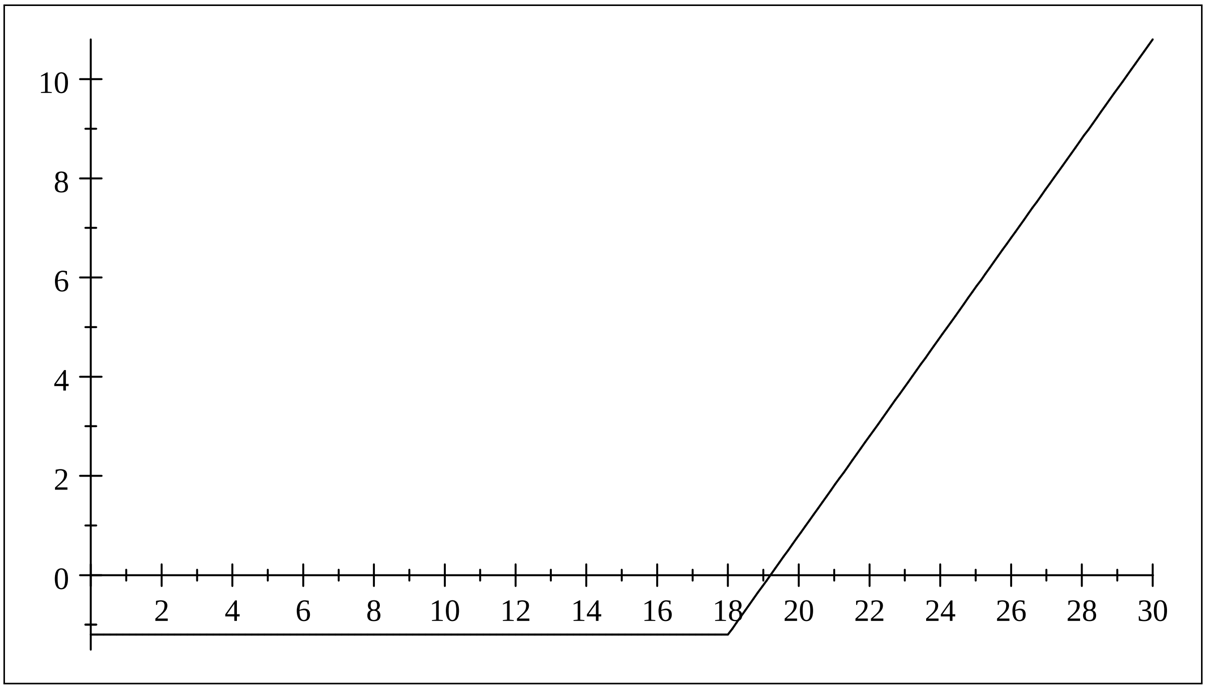
Options

- Call option ("call"): A call option gives the holder the right (but not the obligation) to buy the underlying asset by a certain date T for a certain price K.
- Put option ("put"): A put option gives the holder the right (but not the obligation) to sell the underlying asset by a certain date T for a certain price K.
- The price K in the contract is known as the exercise price or strike price.
- The date T in the contract is known as the expiration date or maturity.
- American options can be exercised at any time up to the expiration date.
- European options can be exercised only on the expiration date itself.
- Long position: investor that buys the option
- Short position: investor that sells the option (writer of the option)

Options

Example: Investor buys 200 contracts of call options - each one over 1 share of AXA with strike $K = 18$ € and maturity T =1 year. Consider that the price of each option ("premium") is 1.2€.
Initial investment= 240€.
If on maturity T the share price is $S_T < 18$ €, the investor has a loss of 240 euros.
If $S_T > 18$, the investor exercises the option. For instance, if $S_T = 25$ €, the total profit is $(25 - 18) \times 200 - 240 = 1160$ €.

Options

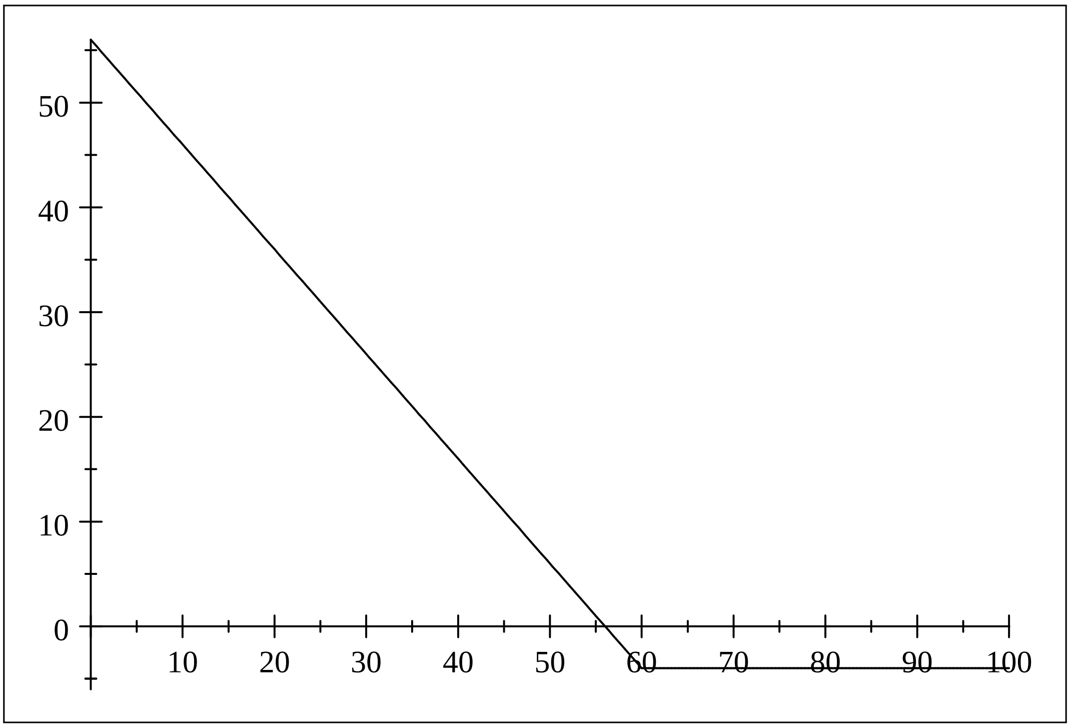


Profit vs. S_T (call option over 1 share)

Options

Example: Investor buys 100 put options - each one over 1 share of BNP Paribas with strike $K = 60$ € and maturity $T = 6$ months. Consider that the price of an option on one share is 4€.
Initial investment = 400 €.
If on maturity T the share price is $S_T < 60$ €, the investor exercises the options. For instance, if $S_T = 50$ €, the total profit is
$(60 - 50) \times 100 - 400 = 600$ €.
If $S_T > 60$ €, the investor does not exercise the options and has a loss of 400 euros.

Options



Profit vs. S_T (put option on 1 share)

- Long position on a "call":

$$\text{Payoff} = f(S_T) = \max(S_T - K, 0)$$

- Short position on a "call"

$$\text{Payoff} = f(S_T) = -\max(S_T - K, 0) = \min(K - S_T, 0)$$

- Long postion on a "put"

$$\text{Payoff} = f(S_T) = \max(K - S_T, 0)$$

- Short position on a "put"

$$\text{Payoff} = f(S_T) = -\max(K - S_T, 0) = \min(S_T - K, 0)$$

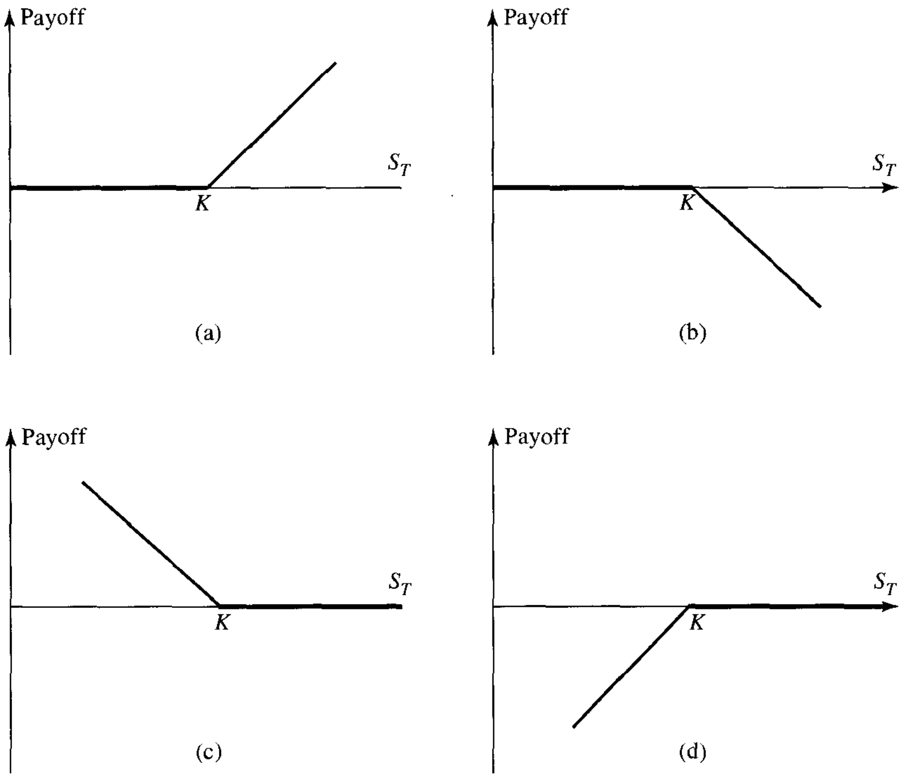


Figura: (a) "Call"longa; (b) "Call"curta; (c) "Put"longa; (d) "Put"curta

Examples

Example (speculation)
A speculator "bets"on the rise of AXA shares prices on the next semester. Assume that today share price ("spot"price) is $S_0 = 18$ €. A "call"option on a share, with strike $K = 22$ € and maturity $T = 6$ months costs 0.50 Euros. Assume that the investor has 1800 € to invest, and consider two future scenarios and two strategies:

Strategy	$S_T = 12$	$S_T = 25$
Buy 100 shares	-600	700
Buy 3600 "call"options	-1800	$10800 - 1800 = 9000$

The options "amplify"the consequences $\Longrightarrow$leverage.

Example (hedging)
In May 2000, an investor has in his portfolio 1000 shares of Microsoft with price $S_0 = 73$ USD/share. The investor is worried that the "antitrust"politics of the US government causes a big downfall of the share price over the next 2 month period. In order to hedge this risk, he buys put options with maturity on July, Strike $K = 65$ USD and "premium"2.5 USD. The cost of this hedge is 2500 USD but he ensures a minimum total value for his portfolio of $65000 - 2500 = 62500$ USD.

- Options are equivalent to have an insurance against adverse movements in the marketand allow to benefit from the favorable movements.

Dangers

- Some times, investors that should be hedgers or arbitrageurs behave like speculators...$\Longrightarrow$Danger!
- Barings Bank (Nick Leeson)
- Subprime crisis
- It is essential to have controls/regulations in order to ensure that derivatives do not put the financial system in danger...

Notation

- t is the current time
- S_t is the underlying share price at time t
- K is the strike or exercise price
- T is the option expiry date
- c_t is the price at time t of a European call option
- p_t is the price at time t of a European put option
- C_t is the price at time t of an American call option
- P_t is the price at time t of an American put option
- r is the risk-free continuously-compounding rate of interest (assumed constant)

Terminology

A call option is:

- in-the-money if the current price, $S_t > K$.
- out-of-the money if $S_t < K$.
- at-the-money if $S_t = K$.

A put option is:

- in-the-money if $S_t < K$.
- out-of-the money if $S_t > K$.
- at-the-money if $S_t = K$.

Terminology

- The intrinsic value of a derivative is the value assuming expiry of the contract immediately rather than at some time in the future. For a call option, for example, the intrinsic value at time t is simply $\max\{S_t - K, 0\}$.
- The intrinsic value of an option is: (i) positive if it is in-the-money, (ii) zero if it is at-the-money or out-of-the-money.
- Question: What is the intrinsic value of a put option at time t?
- Question: Why is the put option exercised only if $S_t < K$?
- The time value or option value of a derivative is the excess of an option premium (current price of the option c_t) over its intrinsic value.

Terminology

- Model for prices: stochastic process S_t adapted to a filtration $\{\mathcal{F}_t, t \geq 0\}$ (that is, given $\mathcal{F}_t$ we know the value ofS_u for all $u \leq t$).
- $\mathcal{F}_t$ is a subset of a larger possible range of past and future events, $\mathcal{F}$.
- Let A be some event contained in $\mathcal{F}$. Then $P(A)$ is the actual "real world" probability that the event A will occur.
- Suppose also that we have a risk-free cash bond which has a value at time t of B_t (we assume that the risk-free rate of interest is constant: that is, $B_t = B_0 e^{rt}$).

Self-financing strategies

- Suppose that at time t we hold the portfolio (φ_t, ψ_t) where φ_t represents the number of units of S_t held at time t and ψ_t is the number of units of the cash bond held at time t.
- Assume that (φ_t, ψ_t) are previsible (φ_t, ψ_t are known based upon the information up to but not including time t).
- Value of the portfolio at time t: $V(t) = \varphi_t S_t + \psi_t B_t$.
- Consider the instantaneous pure investment gain in the value of this portfolio over the period t up to $t + dt$ (assuming that there is no inflow or outflow of cash during the period $[t, t + dt]$). This is equal to $\varphi_t dS_t + \psi_t dB_t$.
- The instantaneous change in the value of the portfolio, allowing for cash inflows and outflows, is given by $dV(t) = \varphi_t dS_t + \psi_t dB_t + S_t d\varphi_t + d\varphi_t.dS_t + B_t d\psi_t + dB_t d\psi_t$.
- The portfolio strategy is self-financing if $dV(t) = \varphi_t dS_t + \psi_t dB_t$.

Complete markets

- Let X be any derivative payment (payoff) contingent upon $\mathcal{F}_U$ where U is the time of payment of X (i.e. X is $\mathcal{F}_U$-measurable and, therefore, depends upon the path of S_t up to time U. The time U may be fixed or it may be a random stopping time.
- A replicating strategy is a self-financing strategy (φ_t, ψ_t), define for $0 \leq t < U$, such that $V(U) = \varphi_U S_U + \psi_U B_U = X$.
- In other words, for an initial investment of $V(0)$ at time 0, if we follow the self-financing portfolio strategy (φ_t, ψ_t) we will be able to reproduce the
- derivative payment without risk.
- The market is said to be complete if for any such contingent claim X there is a replicating strategy (φ_t, ψ_t).

Factors affecting option prices

- A number of mathematical models are used to value options. One of the more widely used is the Black-Scholes model. This uses five parameters to value an option on a nondividend-paying share. The five parameters are:
(1) Underlying share price: S_t
(2) The strike price: K
(3) The time to expiry: $T - t$
(4) The volatility of the underlying asset: σ
(5) The risk-free interest rate: r

Share price

- Note that the price for a call option is always greater than the intrinsic value (this follows on from the lower bound derived later):

$$c_t \geq S_t - Ke^{-r(T-t)} > S_t - K.$$

- Call option: a higher share price means a higher intrinsic value. A higher intrinsic value means a higher premium.
- Put option: a higher share price will mean a lower intrinsic value and a lower premium.

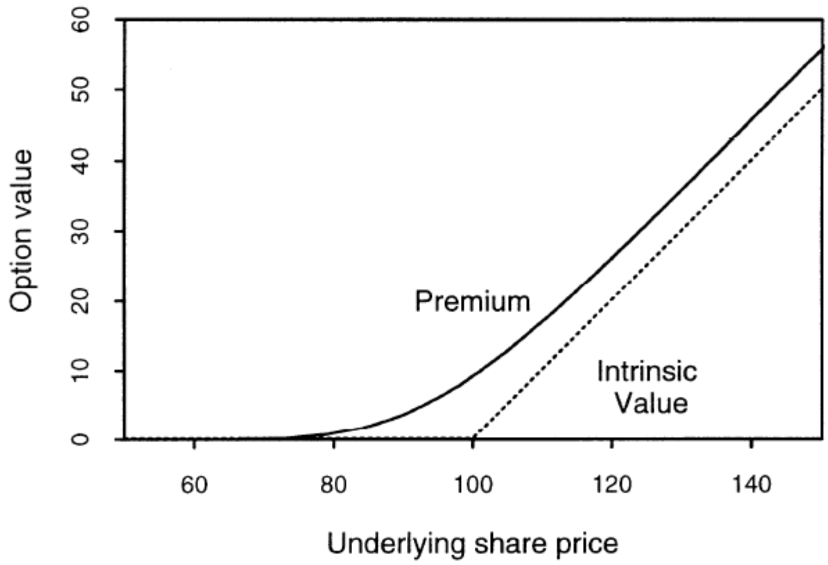


Figure 1: Call option premium and intrinsic value as a function of the current share price, S_t

Strike price

- Call option: a higher strike price means a lower intrinsic value. A lower intrinsic value means a lower premium.
- Put option: a higher strike price will mean a higher intrinsic value and a higher premium.

Time to expiry

- The longer the time to expiry, the greater the chance that the underlying share price can move significantly in favour of the holder of the option before expiry. So the value of an option will increase with term to maturity.
- This increase is moderated slight by the change in the time value of money.

Volatility

- The higher the volatility of the underlying share, the greater the chance that the underlying share price can move significantly in favour of the holder of the option before expiry. So the value of an option will increase with the volatility of the underlying share.

Interest rates

- Call option: an increase in the risk-free rate of interest will result in a higher value for the option because the money saved by purchasing the option rather than the underlying share can be invested at this higher rate of interest, thus increasing the value of the option.
- Put option: higher interest means a lower value (put options can be purchased as a way of deferring the sale of a share: the money is tied up for longer)

Dividend Income

- The basic Black-Scholes model can be adapted to allow for a sixth factor determining the value of an option: dividend income
- Normally the dividend income is not passed onto the holder of an option.
- Call option: the higher the level of dividend income received, the lower is the value of a call option, because by buying the option instead of the underlying share the investor loses this income.
- Put option: the higher the level of dividend income received, the higher is the value of a put option, because buying the option is a way of deferring the sale of a share and the dividend income is received.

- Question: The longer the time to expiry, the greater the chance that the underlying share price can move significantly against the holder of the option before expiry. Why therefore does the value of a call option increase with term to maturity?
- Question: List the 5 factors that determine the price of an american put option and, for each factor, state wether an increase in its value produces an increase or a decrease in the value of the option.

- We can calculate the partial derivatives of the option price with respect to each of the above factors (this quantifies the change of the price in response to a change in each factor): these derivatives are known as the "greeks"and are named after greek letters. We will discuss these "greeks"later.