**Probability Theory and Stochastic Processes**

**LIST 1**[1]

**Measure and probability**

(1) Decide if $\mathcal{F}$ is $\sigma$-algebra of $\Omega$, where:
   (a) $\mathcal{F} = \mathcal{P}(\Omega)$, $\Omega = \mathbb{R}^n$.
   (b) $\mathcal{F} = \{\emptyset, \{1,2\}, \{3,4,5,6\}, \Omega\}$, $\Omega = \{1,2,3,4,5,6\}$.
   (c) $\mathcal{F} = \big\{\emptyset, \{0\}, \mathbb{R}^-, \mathbb{R}_0^-, \mathbb{R}^+, \mathbb{R}_0^+, \mathbb{R} \setminus \{0\}, \mathbb{R}\big\}$, $\Omega = \mathbb{R}$.

(2) Let $(\Omega, \mathcal{F})$ be a measurable space and $A_1, A_2, \dots \in \mathcal{F}$. Prove:
   (a) $\bigcap_{i=1}^{+\infty} A_i \in \mathcal{F}$
   (b) $A_1 \setminus A_2 \in \mathcal{F}$

(3) Let $\Omega$ be a finite set with $\#\Omega = n$. Compute $\#\mathcal{P}(\Omega)$. *Hint*: Find a bijection between $\mathcal{P}(\Omega)$ and the space $\{v \in \mathbb{R}^n \colon v_i \in \{0,1\}\}$.

(4) Determine if the intersection and the union of $\sigma$-algebras are still $\sigma$-algebras.

(5) Let $\Omega = [-1, 1] \subset \mathbb{R}$. Determine if the following collection of sets is a $\sigma$-algebra:

$$\mathcal{F} = \{A \in \mathcal{B}(\Omega) \colon x \in A \Rightarrow -x \in A\}.$$

(6) Show that
   (a) if $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{P}$, then $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2)$.
   (b) $\sigma(\sigma(\mathcal{A})) = \sigma(\mathcal{A})$ for any $\mathcal{A} \subset \mathcal{P}$.
   (c) $\mathcal{A} = \{[a, +\infty[\colon a \in \mathbb{R}\}$ generates the Borel $\sigma$-algebra of $\mathbb{R}$.

(7) Let $\mu \colon \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ be given by

$$\mu(\emptyset) = 0, \quad \mu(\mathbb{R}) = 2, \quad \mu(X) = 1 \quad \text{se} \quad X \in \mathcal{P}(\mathbb{R}) \setminus \{\emptyset, \mathbb{R}\}.$$

Determine if $\mu$ is $\sigma$-subadditive and $\sigma$-additive.

(8) Prove that if $\mu_1, \mu_2$ are measures and $\alpha, \beta \geq 0$, then $\mu = \alpha\mu_1 + \beta\mu_2$ is also a measure.

[1]Send comments and/or corrections to jldias@iseg.utl.pt. Harder questions are marked with *. Collaboration among colleagues is encouraged, but each student should write his/her own solutions, understand them and give credit to the collaborators.

(9) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $A_1, A_2, \dots \in \mathcal{F}$. Prove that:

(a) In the definition of measure, the condition $\mu(\emptyset) = 0$ can be replaced by the existence of a set $E \in \mathcal{F}$ with finite measure, $\mu(E) < +\infty$.

(b) If $A_i \subset A_{i+1}$, then $\mu(\bigcup_i A_i) = \lim_{i \to +\infty} \mu(A_i)$.

(c) If $A_{i+1} \subset A_i$ and $\mu(A_1) < +\infty$, then $\mu(\bigcap_i A_i) = \lim_{i \to +\infty} \mu(A_i)$.

(10) Let $(\Omega, \mathcal{F}, P)$ probability space, $A_1, A_2, \dots \in \mathcal{F}$ and $B$ is the set of points in $\Omega$ that belong to an infinite number of $A_n$'s:

$$B = \bigcap_{n=1}^{+\infty} \bigcup_{k=n}^{+\infty} A_k.$$

Show that:

(a) (First Borel-Cantelli lemma) If

$$\sum_{n=1}^{+\infty} P(A_n) < +\infty,$$

then $P(B) = 0$.

(b) *(Second Borel-Cantelli lemma) If

$$\sum_{n=1}^{+\infty} P(A_n) = +\infty$$

and

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i),$$

for every $n \in \mathbb{N}$ (i.e. the events are mutually independent), then $P(B) = 1$.