Stochastic integration and Itô formula

João Guerra

CEMAPRE and ISEG, UTL

November 4, 2014

- The stochastic integral $I_t(F) = \int_0^t \int_E F(s, x) M(ds, dx)$, with $F \in \mathcal{H}_2$, satisfies:

  1. I_t is a linear operator
  2. $\mathbb{E}[I_t(F)] = 0$;
     $E\left[(I_t(F))^2\right] = \int_0^t \int_{E-\{0\}} \mathbb{E}\left[|F(s, x)|^2\right] \nu(dx)\, ds + \delta_0(E) \int_0^t \mathbb{E}\left[|F(s, 0)|^2\right]\, ds.$
  3. $\{I_t(F), t \geq 0\}$ is $\{\mathcal{F}_t\}$ adapted
  4. $\{I_t(F),\ t \geq 0\}$ is a square-integrable martingale.

Sketch of the Proof of (3): Let $(F_n, n \in \mathbb{N})$ be a sequence of simple processes in $\mathcal{H}_2$ converging to F.
Then $(I_t(F_n), t \geq 0)$ is adapted and $I_t(F_n) \longrightarrow I_t(F)$ in L^2.
Therefore, there is a subsequence $(F_{n_k}; n_k \in \mathbb{N})$ such that $I_t(F_{n_k}) \longrightarrow I_t(F)$ a.s. as $n_k \to \infty$.

Therefore $\{I_t(F), \ t \geq 0\}$ is $\{\mathcal{F}_t\}$ adapted. ∎

Sketch of the Proof of (4):
Let F be a simple process in $\mathcal{H}_2$ and choose $0 < s = t_l < t_{l+1} < t$.

Then $I_t(F) = I_s(F) + I_{s,t}(F)$ and by prop. (3),

$$\mathbb{E}_s(I_t(F)) = I_s(F) + \mathbb{E}_s(I_{s,t}(F))$$

Moreover,

$$\mathbb{E}_s(I_{s,t}(F)) = \mathbb{E}_s \left(\sum_{j=l+1}^{m} \sum_{k=1}^{n} F_k(t_j) M((t_j, t_{j+1}], A_k) \right)$$

$$= \sum_{j=l+1}^{m} \sum_{k=1}^{n} \mathbb{E}_s(F_k(t_j)) \mathbb{E}_s[M((t_j, t_{j+1}], A_k)] = 0.$$

Therefore $\mathbb{E}_s(I_t(F)) = I_s(F)$ and $\{I_t(F), \ t \geq 0\}$ is a martingale.
Now, let $(F_n, n \in \mathbb{N})$ be a sequence of simple processes converging to F in L^2.
It can be proved that (see Applebaum) $\mathbb{E}_s(I_t(F_n)) \to \mathbb{E}_s(I_t(F))$ in L^2 and therefore $\mathbb{E}_s(I_t(F)) = I_s(F)$ is a square-integrable martingale∎

Lévy type stochastic integrals

- We say Y is a Lévy type stochastic integral if

$$Y_t^i = Y_0 + \int_0^t G^i(s)\, ds + \int_0^t F_j^i(s)\, dB_s^j + \int_0^t \int_{|x|<1} H^i(s,x)\, \widetilde{N}(ds, dx)$$

$$+ \int_0^t \int_{|x| \geq 1} K^i(s,x)\, N(ds, dx), \quad i = 1, ..., d, j = 1, ..., m \qquad (1)$$

where G^i, F_j^i, H^i and K are processes such that the integrals are well defined.

- With stochastic differentials notation, in the one-dimensional case, we can write:

$$dY(t) = G(t)\, dt + F(t)\, dB(t) + \int_{|x|<1} H(t,x)\, \widetilde{N}(dt, dx)$$

$$+ \int_{|x| \geq 1} K(t,x)\, N(dt, dx).$$

Lévy type stochastic integrals

- Let M be an adapted and left-continuous process. Then we can define a new process $\{Z_t, t \geq 0\}$ by

$$dZ(t) = M(t)\, dY(t)$$

or

$$dZ(t) = M(t)\, G(t)\, dt + M(t)\, F(t)\, dB(t) + M(t)\, H(t,x)\, \widetilde{N}(dt, dx)$$
$$+ M(t)\, K(t,x)\, N(dt, dx).$$

Example - Lévy stochastic integrals

- X: Lévy process with characteristics (b, A, ν) and Lévy-Itô decomposition

$$X(t) = bt + B_A(t) + \int_{|x|<1} x\widetilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx).$$

  Let $U \in \mathcal{H}_2(t)$ for all $t \geq 0$. and choose in (1) $F_j^i = A_j^i U$, $H^i = K^i = x^i U$.
- The process Y such that

$$dY(t) = U(t)\, dX(t)$$

  is called a Lévy stochastic integral.

Example - Ornstein Uhlenbeck (OU) process

- OU process:

$$Y(t) = e^{-\lambda t} y_0 + \int_0^t e^{-\lambda(t-s)} dX(s),$$

  where y_0 is fixed.
- This process can be used for volatility modelling in finance.
- Exercise: Prove that if X is a one-dimensional Brownian motion then $Y(t)$ is a Gaussian process with mean $e^{-\lambda t} y_0$ and variance $\frac{1}{2\lambda}\left(1 - e^{-2\lambda t}\right)$

Example - Ornstein Uhlenbeck (OU) process

- In differential form, the OU process is the solution of the SDE:

$$dY(t) = -\lambda Y(t)\, dt + dX(t),$$

  which is known as the Langevin equation (is a stochastic differential equation).

- The Langevin equation is also a model for the physical phenomenon of Brownian motion: includes the viscous drag of the medium on the particle as well as random fluctuations.

Itô formula for Poisson stochastic integrals

- Consider the Poisson stoch. integral
  $W(t) = W(0) + \int_0^t \int_A K(s,x)\, N(ds, dx)$, with A bounded below and K predictable.

Lemma

(Itô formula 1): If $f \in C(\mathbb{R})$ then

$$f(W(t)) - f(W(0)) = \int_0^t \int_A [f(W(s-) + K(s,x)) - f(W(s-))]\, N(ds, dx) \quad a.s.$$

Itô formula for Poisson stochastic integrals

Proof: Let $Y(t) = \int_A x N(t, dx)$. The jump times of Y can be defined by $T_0^A = 0$, $T_n^A = \inf\left\{t > T_{n-1}^A; \Delta Y(t) \in A\right\}$
Then

$$f(W(t)) - f(W(0)) = \sum_{0 \leq s \leq t} [f(W(s)) - f(W(s-))]$$

$$= \sum_{n=1}^{\infty} \left[f\left(W\left(t \wedge T_n^A\right)\right) - f\left(W\left(t \wedge T_{n-1}^A\right)\right)\right]$$

$$= \sum_{n=1}^{\infty} f\left(W\left(t \wedge T_n^A -\right) + K\left(t \wedge T_n^A, \Delta Y\left(t \wedge T_n^A\right)\right)\right) - f\left(W\left(t \wedge T_n^A -\right)\right)$$

$$= \int_0^t \int_A [f(W(s-) + K(s, x)) - f(W(s-))] N(ds, dx).$$

Itô formula for Brownian motion

- Let M be a Brownian integral with drift:

$$M^i(t) = \int_0^t F_j^i(s) \, dB^j(s) + \int_0^t G^i(s) \, ds.$$

- Let us define the quadratic variation process:

$$\left[M^i, M^j\right](t) = \sum_{k=1}^{m} \int_0^t F_k^i(s) F_k^j(s) \, ds.$$

Itô formula for Brownian motion

Theorem

(Itô formula 2) If $f \in C^2 \left(\mathbb{R}^d \right)$ then

$$f \left(M \left(t \right) \right) - f \left(M \left(0 \right) \right) = \int_0^t \partial_i f \left(M \left(s \right) \right) dM^i \left(s \right) + \frac{1}{2} \int_0^t \partial_i \partial_j f \left(M \left(s \right) \right) d \left[M^i, M^j \right] \left(s \right). \quad a.s.$$

Proof: See Applebaum

Itô formula for Lévy type stochastic integrals

- Let

$$dY(t) = G(t)\, dt + F(t)\, dB(t) + \int_{|x|<1} H(t,x)\, \widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\, N(dt,dx)$$

- $dY_c(t) := G(t)\, dt + F(t)\, dB(t)$
- $dY_d(t) := \int_{|x|<1} H(t,x)\, \widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\, N(dt,dx)$

Itô formula for Lévy type stochastic integrals

Theorem

(Itô formula 3): If $f \in C^2\left(\mathbb{R}^d\right)$ then

$$f\left(Y\left(t\right)\right) - f\left(Y\left(0\right)\right) = \int_0^t \partial_i f\left(Y\left(s-\right)\right) dY_c^i\left(s\right) + \frac{1}{2}\int_0^t \partial_i\partial_j f\left(Y\left(s-\right)\right) d\left[Y_c^i, Y_c^j\right]\left(s\right)$$

$$+ \int_0^t \int_{|x|\geq 1} \left[f\left(Y\left(s-\right) + K\left(s, x\right)\right) - f\left(Y\left(s-\right)\right)\right] N\left(ds, dx\right)$$

$$+ \int_0^t \int_{|x|<1} \left[f\left(Y\left(s-\right) + H\left(s, x\right)\right) - f\left(Y\left(s-\right)\right)\right] \widetilde{N}\left(ds, dx\right)$$

$$+ \int_0^t \int_{|x|<1} \left[f\left(Y\left(s-\right) + H\left(s, x\right)\right) - f\left(Y\left(s-\right)\right)\right.$$

$$\left. - H^i\left(s, x\right) \partial_i f\left(Y\left(s-\right)\right)\right] \nu\left(dx\right) ds$$

- Proof: see Applebaum

Itô formula for Lévy type stochastic integrals

Theorem

(Itô formula 4): If $f \in C^2\left(\mathbb{R}^d\right)$ then

$$f\left(Y\left(t\right)\right) - f\left(Y\left(0\right)\right) = \int_0^t \partial_i f\left(Y\left(s-\right)\right) dY^i\left(s\right) + \frac{1}{2}\int_0^t \partial_i\partial_j f\left(Y\left(s-\right)\right) d\left[Y_c^i, Y_c^j\right]\left(s\right)$$

$$+ \sum_{0\leq s\leq t} \left[f\left(Y\left(s\right)\right) - f\left(Y\left(s-\right)\right) - \Delta Y^i\left(s\right) \partial_i f\left(Y\left(s-\right)\right)\right].$$

- Proof: see Applebaum

Itô formula for Lévy type stochastic integrals

- Quadratic variation process for Y:

$$\left[Y^i, Y^j\right](t) = \left[Y_c^i, Y_c^j\right](t) + \sum_{0 \le s \le t} \Delta Y^i(s) \Delta Y^j(s).$$

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$$\left[Y^i, Y^j\right](t) = \sum_{k=1}^{m} \int_0^t F_k^i(s) F_k^j(s) \, ds + \int_0^t \int_{|x|<1} H^i(s,x) H^j(s,x) \widetilde{N}(ds,dx)$$

$$\tag{2}$$

$$+ \int_0^t \int_{|x|\ge 1} K^i(s,x) K^j(s,x) N(ds,dx).$$

Itô's product formula

Theorem

If Y^1 and Y^2 are real valued Lévy type stochastic integrals then

$$Y^1(t) Y^2(t) = Y^1(0) Y^2(0) + \int_0^t Y^1(s-) \, dY^2(s)$$

$$+ \int_0^t Y^2(s-) \, dY^1(s) + \left[Y^1, Y^2\right](t).$$

Proof Take $f(x_1, x_2) = x_1 x_2$ and apply Itô's formula 4:

$$Y^1(t) Y^2(t) - Y^1(0) Y^2(0) = \int_0^t Y^1(s-) \, dY^2(s)$$

$$+ \int_0^t Y^2(s-) \, dY^1(s) + \left[Y_c^1, Y_c^2\right](t)$$

$$+ \sum_{0 \le s \le t} \left[Y^1(s) Y^2(s) - Y^1(s-) Y^2(s-) - \Delta Y^1(s) Y^2(s-) - \Delta Y^2(s) Y^1(s-)\right]$$

and the result follows.

Itô's product formula

- Product formula in differential form:

$$d\left(Y^1\left(t\right)Y^2\left(t\right)\right) = Y^1\left(t-\right)dY^2\left(t\right) + Y^2\left(t-\right)dY^1\left(t\right) + d\left[Y^1, Y^2\right]\left(t\right).$$

- The Itô correction arises as the result of the following formal product relations (see (2)):

$$dB^i\left(t\right)dB^j\left(t\right) = \delta^{ij}dt,$$
$$N\left(dt, dx\right)N\left(dt, dy\right) = N\left(dt, dx\right)\delta\left(x - y\right),$$
all other products of differentials vanish.

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