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Processes of normal inverse Gaussian type

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Abstract. With the aim of modelling key stylized features of observational series from finance and turbulence a number of stochastic processes with normal inverse Gaussian marginals and various types of dependence structures are discussed. Ornstein-Uhlenbeck type processes, superpositions of such processes and stochastic volatility models in one and more dimensions are considered in particular, and some discussion is given of the feasibility of making likelihood inference for these models.

Key words: Background driving Lévy processes, long range dependence, Ornstein-Uhlenbeck type, selfdecomposability, stochastic volatility

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1 Introduction

In the field of finance, distributions of logarithmic asset returns can often be fitted extremely well by normal inverse Gaussian distributions (Barndorff-Nielsen 1995, 1996a; Rydberg 1996a,b; see also Rydberg 1997). It is therefore of some interest, using normal inverse Gaussian laws as building blocks, to construct stochastic process models for stock prices and asset returns that capture as many as possible of the key stylised features of financial time series.

Normal inverse Gaussian distributions have also considerable potential with respect to modelling in quite different contexts. This is true, in particular, in turbulence where data on velocities and velocity differences from high Reynolds

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number wind fields show features that are remarkably similar to stylised facts in finance; cf. Barndorff-Nielsen (1996b) and Subsect. 2.0 below.

With these areas of application in mind, particularly those of finance, we study in the present paper several kinds of processes with normal inverse Gaussian marginals, in particular processes of Ornstein — Uhlenbeck type, superpositions of such processes and stochastic volatility models in one and several dimensions. An important aim is to find models with analytically and statistically tractable correlation structure. The feasibility of carrying out likelihood inference under the derived models is discussed to some extent. Models of diffusion type will not be considered here due to the difficulty of handling temporal dependence structures under such models.

Section 2 begins with some considerations of goals for the modelling; further, the section contains various preliminary, mostly well known, results concerning: normal inverse Gaussian distributions; infinite divisibility and exponential families; selfdecomposability; Ornstein - Uhlenbeck type processes and their background driving Lévy processes (BDLP); long range dependence and self-similarity; and, finally, processes of type G. In Sect. 3 we then define the normal inverse Gaussian Ornstein - Uhlenbeck process and we characterize the associated BDLP as a sum of three homogeneous Lévy processes, one being the normal inverse Gaussian Lévy process and the second a compound Poisson process (the third process is present only in case the one-dimensional marginal distributions are asymmetric). An inverse Gaussian Ornstein-Uhlenbeck process is also considered. Normal inverse Gaussian processes with (quasi) long range dependence are constructed in Sect. 4, and Sect. 5 contains a brief discussion of a selfsimilar inverse Gaussian process with independent increments. Section 6 discusses modelling by superposition of Ornstein - Uhlenbeck type processes, partly as a preparation for the final Sect. 7 which concerns stochastic volatility models, in one and more dimensions. As already indicated, questions of likelihood based analysis are discussed for several of the models.

2 Background

This section reviews a number of – mostly well known – results, on normal inverse Gaussian distributions, infinite divisibility and exponential families, self-decomposability, Ornstein-Uhlenbeck type processes, long range dependence and selfsimilarity, and processes of type G. These results are needed for the construction and study of the normal inverse Gaussian processes to be discussed in Sects. 3 to 7. First, however, some issues of statistical modelling, relating in particular to the analysis of financial data, are discussed.

2.0 Modelling considerations

A number of characteristic features of observational series from finance and from turbulence are summarised in Table 1. The features are widely recognized as being esssential for understanding and modelling within these two, quite different, subject areas. In finance the observational series concerned consist of values of assets such as stocks or (logarithmic) stock returns or exchange rates, while in turbulence the series typically give the velocities or velocity derivatives (or differences), in the mean wind direction of a large Reynolds number wind field. For some typical examples of empirical probability densities of velocity differences in large Reynolds number wind fields see, for instance, Barndorff-Nielsen (1996b).

Table 1. Stylised features

	Finance	Turbulence
semiheavy tails	+	+
asymmetry	(+)	+
varying activity	volatility	intermittency
aggregational Gaussianity	+	+
quasi long range dependence	+	(+)
scaling/selfsimilarity	+	+

The term ‘semiheavy tails’, in Table 1, is intended to indicate that the data suggest modelling by probability distributions whose densities behave, for $x \to \pm\infty$, as

$$const. \, |x|^{\rho_\pm} \exp(-\sigma_\pm \, |x|)$$

for some $\rho_+, \rho_- \in \mathbf{R}$ and $\sigma_+, \sigma_- > 0$.

Distributions of financial asset returns are generally rather close to being symmetric around 0, but there is a definite tendency towards asymmetry stemming from the fact that the market is prone to react differently to positive as opposed to negative returns, cf. for instance Shephard (1996; Subsect. 1.3.4). Velocity differences in turbulence show an inherent asymmetry consistent with Kolmogorov’s modified theory of homogeneous high Reynolds number turbulence (cf. Barndorff-Nielsen 1986).

A very characteristic trait of time series from turbulence as well as finance is that there seems to be a kind of switching regime between periods of relatively small random fluctuations and periods of high ‘activity’. In turbulence this phenomenon is known as intermittency, see e.g. Frisch (1995; chapter 8) for a thorough discussion, whereas in finance one speaks of stochastic volatility or conditional heteroscedasticity.

By aggregational Gaussianity is meant the fact that long term aggregation of financial asset returns, in the sense of summing the returns over longer periods, will lead to approximately normally distributed variates, and similarly in the turbulence context; i.e. a normal central limit effect rules. For illustrations of this, see for instance Eberlein and Keller (1995) and Barndorff-Nielsen (1996b).

Finally in Table 1 we refer, by ‘quasi long range dependence’, to the patterns of autocorrelations so typically observed in time series from both of the two fields of investigation and which indicate dependence structures close to what is defined mathematically as ‘long range dependence’. Methods for modelling dependence behaviour of this kind are discussed in the Sects. 4, 6 and 7.

2.1 NIG distributions

The normal inverse Gaussian distribution with parameters α, β, μ and δ is denoted $NIG(\alpha, \beta, \mu, \delta)$ and it may be defined as follows. Consider a bivariate Brownian motion (u_t, v_t) starting at the point $(\mu, 0)$ and having constant drift vector (β, γ) with $\gamma > 0$, and let z denote the time at which the second component v_t hits the line $v = \delta > 0$ for the first time. (The coordinate processes u_t and v_t are assumed to be independent.) Then, letting $\alpha = \sqrt{(\beta^2+\gamma^2)}$, the law of u_z is $NIG(\alpha, \beta, \mu, \delta)$. Equivalently, $NIG(\alpha, \beta, \mu, \delta)$ may be described as the distribution at time $t = 1$ of x_t where

$$x_t = u_{z_t} + \mu t, \tag{2.1}$$

with z_t, assumed independent of the process u_t, being the inverse Gaussian Lévy process. The latter process is defined as the homogeneous Lévy process for which $z_1 \overset{d}{=} z$, where $\overset{d}{=}$ means 'distributed as' and where the distribution of z is the inverse Gaussian law whose probability density function is given by

$$(2\pi)^{-1/2}\delta e^{\delta\gamma} z^{-3/2} \exp\{-(\delta^2 z^{-1} + \gamma^2 z)/2\} .$$

This distribution is denoted $IG(\delta, \gamma)$. The process x_t is also a homogeneous Lévy process, termed the normal inverse Gaussian Lévy process.

The density function of the $NIG(\alpha, \beta, \mu, \delta)$ distribution is

$$g(x; \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta) q\left(\frac{x-\mu}{\delta}\right)^{-1} K_1\left\{\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right\} e^{\beta x} \tag{2.2}$$

where $q(x) = \sqrt{(1+x^2)}$ and

$$a(\alpha, \beta, \mu, \delta) = \pi^{-1}\alpha e^{\delta\sqrt{(\alpha^2-\beta^2)}-\beta\mu} \tag{2.3}$$

and where K_1 is the modified Bessel function of third order and index 1.

It follows immediately from (2.2) and (2.3) that the moment generating function of the normal inverse Gaussian distribution is

$$M(u; \alpha, \beta, \mu, \delta) = \exp[\delta\{\sqrt{(\alpha^2 - \beta^2)} - \sqrt{(\alpha^2 - (\beta+u)^2)}\} + \mu u] . \tag{2.4}$$

Thus, in particular, if $x_1, \ldots, x_m$ are independent normal inverse Gaussian random variables with common parameters α and β but having individual location-scale parameters μ_i and $\delta_i (i = 1, \ldots, m)$ then $x_+ = x_1 + \ldots + x_m$ is again distributed according to a normal inverse Gaussian law, with parameters $(\alpha, \beta, \mu_+, \delta_+)$.

It is often of interest to consider alternative parametrisations of the normal inverse Gaussian laws. In particular, letting $\overline{\alpha} = \delta\alpha$ and $\overline{\beta} = \delta\beta$, we have that $\overline{\alpha}$ and $\overline{\beta}$ are invariant under location—scale changes, and when $\overline{\alpha}, \overline{\beta}, \mu, \delta$ constitute the parametrisation of interest we shall write $NIG[\overline{\alpha}, \overline{\beta}, \mu, \delta]$ instead of $NIG(\alpha, \beta, \mu, \delta)$. In terms of this alternative parametrisation the mean and variance of $NIG[\overline{\alpha}, \overline{\beta}, \mu, \delta]$ are

$$\kappa_1 = \mu + \delta\rho/(1-\rho^2)^{1/2}$$

and

$$\kappa_2 = \delta^2/\{\bar{\alpha}(1-\rho^2)^{3/2}\}$$

where $\rho = \beta/\alpha$, which is invariant since $\beta/\alpha = \bar{\beta}/\bar{\alpha}$.

We note finally that the NIG distribution (2.2) has semiheavy tails; specifically,

$$g(x;\alpha,\beta,\mu,\delta) \sim const.\,|x|^{-3/2}\,e^{-\alpha|x|+\beta x} \quad \text{as} \quad x \to \pm\infty\ ,$$

as follows from the well known asymptotic relation for the Bessel functions $K_\nu(x)$:

$$K_\nu(x) \sim \sqrt{(\pi/2)}x^{-1/2}e^{-x} \quad \text{as} \quad x \to \infty\ .$$

For further results relating to the normal inverse Gaussian distributions see Barndorff-Nielsen (1995, 1996a,b) and Rydberg (1996a,b, 1997).

2.2 Infinite divisibility and exponential models

We start by recalling a few wellknown facts about infinitely divisible distributions.

A Lévy measure U is a positive Radon measure on $\mathbf{R}\setminus\{0\}$ such that

$$\int_{\mathbf{R}\setminus\{0\}} \min(1,x^2)U(\mathrm{d}x) < \infty\ .$$

Let $\mathcal{U}$ denote the set of all Lévy measures and let

$$\mathcal{U}_0 = \{U \in \mathcal{U}\colon U \text{ is bounded}\}$$

$$\mathcal{U}_1 = \{U \in \mathcal{U}\setminus\mathcal{U}_0\colon \int_{\mathbf{R}\setminus\{0\}} \min(1,|x|)U(\mathrm{d}x) < \infty\}\ .$$

A centering function is a continuous bounded function $\tau : \mathbf{R}\to\mathbf{R}$ such that $x \to \{\tau(x)-x\}/x^2$ is also bounded.

If P is an infinitely divisible probability measure on $\mathbf{R}$ then, given a centering function τ, there exists a triplet (U,σ^2,χ), where U is a Lévy measure, $\sigma^2 \geq 0$ and $\chi \in \mathbf{R}$, such that, letting $\phi(\zeta;\chi,\sigma^2) = \exp(-\frac{1}{2}\sigma^2\zeta^2 + i\chi\zeta)$, we have

$$\text{(2.5)} \quad \int_{-\infty}^{\infty} e^{i\zeta x}P(dx) = \begin{cases} \phi(\zeta;\chi,\sigma^2)\exp\int_{\mathbf{R}\setminus\{0\}}\{e^{i\zeta x}-1\}U(\mathrm{d}x) \\ \qquad \text{if } U \in \mathcal{U}_0\cup\mathcal{U}_1 \\ \phi(\zeta;\chi,\sigma^2)\exp\int_{\mathbf{R}\setminus\{0\}}\{e^{i\zeta x}-1-i\zeta\tau(x)\}U(\mathrm{d}x) \\ \qquad \text{if } U \in \mathcal{U}\setminus\{\mathcal{U}_0\cup\mathcal{U}_1\} \end{cases}$$

Conversely, if (U,σ^2,χ) is a triplet as above then formula (2.5) determines an infinitely divisible probability measure P. The collection (U,σ^2,χ) is termed the *characteristic triplet* of P.

Suppose υ is a nondegenerate Radon measure on $\mathbf{R}$, define

$$D(\upsilon) = \{\theta \in \mathbf{R} : \int e^{\theta x}\upsilon(dx) < \infty\}$$

and let $\Theta(\upsilon) = \text{int}D(\upsilon)$, the interior of $D(\upsilon)$. We denote by M the set of all nondegenerate Radon measures υ such that $\Theta(\upsilon)$ is not empty.

If $\upsilon \in M$, we define the *full natural exponential family* $\bar{F}(\upsilon)$ generated by υ and the identity mapping on $\mathbf{R}$ as the class of probability measures P_θ, $\theta \in D(\upsilon)$, such that $P_\theta << \upsilon$ and

$$p(x;\theta) = \exp\{\theta x - k_\upsilon(\theta)\}$$

where $p(\cdot;\theta) = \mathrm{d}P_\theta/\mathrm{d}\upsilon$ and where $k_\upsilon(\theta)$, the *cumulant function* of the exponential family, is given by $k_\upsilon(\theta) = \log \int e^{\theta x}\upsilon(dx)$. As is well known, the cumulant function is infinitely often differentiable on $\Theta(\upsilon)$. The subfamily $F(\upsilon)$ of $\bar{F}(\upsilon)$ consisting of those P_θ for which $\theta \in \Theta(\upsilon)$ is called the *natural exponential family* generated by υ.

From Bar-Lev et al. (1992) (cf., also, Letac 1992; Sects. 1.5-6) we have

Theorem 2.1. *Let $\upsilon \in M$ and let k_υ be the cumulant function of $\bar{F}(\upsilon)$. Then the following three statements are equivalent:*

(i) There exists an element P of $\bar{F}(\upsilon)$ such that P is infinitely divisible.
(ii) All elements of $\bar{F}(\upsilon)$ are infinitely divisible.
(iii) There exists a V in M such that $\Theta(V) = \Theta(\upsilon)$ and

$$k''_\upsilon(\theta) = \int_{\mathbf{R}} e^{\theta x} V(\mathrm{d}x)\ .$$

Furthermore, for θ in $D(\upsilon)$ the Lévy measure U_θ corresponding to $p(x;\theta)$ is given by

$$U_\theta(x) = x^{-2}\exp(\theta x)[V(\mathrm{d}x) - V(\{0\})\delta_0(\mathrm{d}x)] \tag{2.6}$$

where δ_0 denotes the degenerate probability measure giving mass 1 to the origin of $\mathbf{R}$.

In particular, if $\theta = 0$ and $V(\{0\}) = 0$ we have for $x \neq 0$

$$U(\mathrm{d}x) = x^{-2}V(\mathrm{d}x)\ . \tag{2.7}$$

□

As an illustration of the usefulness of the theorem, we now present a simple derivation of the Lévy measure of the $NIG(\alpha,\beta,\mu,\delta)$ distribution. (The infinite divisibility of $NIG(\alpha,\beta,\mu,\delta)$ is an immediate consequence of formula (2.4).) Suppose first that $\mu = \beta = 0$ in which case the cumulant transform of the distribution is

$$k(\theta) = k(\theta;\alpha,\delta) = \delta\alpha - \delta(\alpha^2 - \theta^2)^{1/2}\ .$$

It follows that

$$k'(\theta) = \delta\theta(\alpha^2 - \theta^2)^{-1/2}$$

$$k''(\theta) = \delta\alpha^2(\alpha^2 - \theta^2)^{-3/2}\ .$$

On the other hand, since

$$e^{k(\theta)} = \int e^{\theta x} g(x; \alpha, 0, 0, \delta) \mathrm{d}x$$

we find, on differentiating twice,

$$\{k''(\theta) + k'(\theta)^2\} e^{k(\theta)} = \int e^{\theta x} x^2 g(x; \alpha, 0, 0, \delta) \mathrm{d}x \ .$$

Dividing through by δ and then letting δ tend to 0 shows, in view of (2.2), that

(2.8) $$k''(\theta; \alpha, 1) = \alpha^2 (\alpha^2 - \theta^2)^{-3/2} = \pi^{-1} \alpha \int |x| K_1(\alpha |x|) e^{\theta x} \mathrm{d}x \ .$$

Hence, by Theorem 2.1, the Lévy measure of $NIG(\alpha, 0, 0, 1)$ has density

$$\pi^{-1} \alpha |x|^{-1} K_1(\alpha |x|) \ .$$

A further direct application of Theorem 2.1 then shows that, for general μ, β and δ, the distribution $NIG(\alpha, \beta, \mu, \delta)$ has characteristic triplet $(F, 0, \chi)$ where F is absolutely continuous with density

(2.9) $$f(x; \alpha, \beta, \delta) = \pi^{-1} \delta \alpha \, |x|^{-1} \, K_1(\alpha \, |x|) e^{\beta x}$$

while

$$\chi = \mu + 2\pi^{-1} \delta \alpha \int_0^1 \sinh(\beta x) K_1(\alpha x) \mathrm{d}x \ .$$

(A much less direct determination of $f(x; \alpha, \beta, \delta)$ and χ was given in Barndorff-Nielsen (1996a).)

As simple consequences of Theorem 2.1 we have

Corollary 2.1. *Suppose that $U(\mathrm{d}x)$ is a Lévy measure whose Laplace transform exists in a neighbourhood of 0, and define the function k on $\Theta(U)$ by*

$$k(\theta) = \int e^{\theta x} U(\mathrm{d}x) \ .$$

Then $k(\theta)$ is the cumulant function of a natural exponential family on **R**, *and the member of this family corresponding to $\theta = 0$ is infinitely divisible with Lévy measure U.* □

Corollary 2.2. *Let $U(\mathrm{d}x)$ be a Lévy measure, let $V(\mathrm{d}x) = x^2 U(\mathrm{d}x)$ for $x \neq 0$ and $V(\{0\}) = 0$, and suppose that the Laplace transform of $V(\mathrm{d}x)$ exists in a neighbourhood of 0. Furthermore, let k be a function on $\Theta(V)$ such that*

$$k''(\theta) = \int e^{\theta x} V(\mathrm{d}x) \ .$$

Then $k(\theta)$ is the cumulant function of a natural exponential family on **R**, *and the member of this family corresponding to $\theta = 0$ is infinitely divisible with Lévy measure U.* □

We shall use these corollaries in Sect. 3.

2.3 Selfdecomposability

A one-dimensional probability measure μ is said to be *selfdecomposable* or to belong to Lévy's class L, if for each $\lambda > 0$ there exists a probability measure ν_λ such that

$$\phi(\zeta) = \phi(e^{-\lambda}\zeta)\phi_\lambda(\zeta)$$

where ϕ and ϕ_λ denote the characteristic function of μ and ν_λ, respectively.

Selfdecomposable distributions are infinitely divisible. On the other hand, we have the following characterization of the class L as a subclass of the set of all infinitely divisible laws (Lukacs 1970).

Lemma 2.1. *Let $U(\mathrm{d}x)$ denote the Lévy measure of an infinitely divisible probability measure P on* $\mathbf{R}$. *Then P is selfdecomposable if and only if U is of the form $U(dx) = u(x)dx$ with*

$$u(x) = |x|^{-1} c(x)$$

where $c(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0,\infty)$.

If u is differentiable then the above necessary and sufficient condition may be reexpressed as

$$u(x) + xu'(x) \leq 0\ . \tag{2.10}$$

□

The following theorem is due to Jurek and Vervaat (1983) (cf. also Jurek and Mason 1993).

Theorem 2.2. *A random variable x has law $\mu \in L$ if and only if x has a representation of the form*

$$x = \int_0^\infty e^{-t}\mathrm{d}z(t)$$

where $z(t)$ is a homogeneous Lévy process.

In this case the Lévy measures U and W of x and $z(1)$ are related by

$$U(dx) = \int_0^\infty W(e^t dx)dt\ . \tag{2.11}$$

□

From (2.11) we find that for $x > 0$

$$\begin{aligned} U([x,\infty)) &= \int_0^\infty W(e^t[x,\infty))\mathrm{d}t \\ &= \int_1^\infty s^{-1}W([sx,\infty))\mathrm{d}s\ . \end{aligned}$$

Provided $W([x,\infty))$ is continuous in x on $(0,\infty)$ this equation may be rewritten as

$$U([x,\infty)) = \int_x^\infty s^{-1} W([s,\infty))\mathrm{d}s \ .$$

A similar conclusion holds, of course, for $U((-\infty,-x])$.

Now suppose that U and W are absolutely continuous with respect to Lebesgue measure, with densities u and w. Then, for $x > 0$,

$$u(x) = x^{-1} W([x,\infty))$$

and it follows that u is differentiable for $x > 0$ and that

(2.12) $$w(x) = -u(x) - xu'(x) \ ;$$

a similar argument for $x < 0$ shows that (2.12) holds for all $x \neq 0$. (Compare with inequality (2.10) in Lemma 2.1.)

2.4 Ornstein-Uhlenbeck processes

A stochastic process $x(t)$ is said to be of Ornstein-Uhlenbeck type if it satisfies a stochastic differential equation of the form

(2.13) $$\mathrm{d}x(t) = -\lambda x(t)\mathrm{d}t + \mathrm{d}z(t) \ .$$

where $z(t)$ is a homogeneous Lévy process, which we refer to as the *background driving Lévy process*, abbreviated (BDLP). For concreteness we take $z(t)$ to be cadlag. Assuming $\lambda > 0$ we have that (2.13) is solved by

(2.14) $$x(t) = e^{-\lambda t}x(0) + \int_0^t e^{-\lambda(t-s)}\mathrm{d}z(s) \ .$$

Let

$$u(t) = \int_0^t e^{-\lambda(t-s)}\mathrm{d}z(s)$$

so that (2.14) takes the form

(2.15) $$x(t) = e^{-\lambda t}x(0) + u(t).$$

Suppose we are given a one-dimensional distribution D, for instance one of the *NIG* distributions, and suppose we wish, for each $\lambda > 0$, to construct an Ornstein-Uhlenbeck type process $x(t)$ that is stationary and such that the law of $x(t)$ is D . Note that when this is possible, D must – in view of (2.15) – be selfdecomposable. Furthermore, each λ has associated with it a BDLP which we denote $z^{(\lambda)}(t)$.

The following theorem, which combines results from Sect. 3 of Barndorff-Nielsen et al. (1995), gives a sufficient condition for the construction. Before stating the theorem we note that if the solution (2.14) is stationary and square integrable and if $\mathrm{E}\{x(0)\} = \mathrm{E}\{z(1)\} = 0$ then the correlation function of $x(t)$ is of the form $\rho(u) = \exp(-\lambda u)$.

Theorem 2.3. *Let $c(\zeta)$ be a differentiable and selfdecomposable characteristic function and let $\kappa(\zeta) = \log c(\zeta)$. Suppose that $\zeta\kappa'(\zeta)$ is continuous at 0 and let $\phi^{(\lambda)}(\zeta) = \lambda\zeta\kappa'(\zeta)$ for a $\lambda > 0$. Then $\exp\{\phi^{(\lambda)}(\zeta)\}$ is an infinitely divisible characteristic function.*

Furthermore, letting $z^{(\lambda)}(t)$ be the homogeneous Lévy process for which $z^{(\lambda)}(1)$ has characteristic function $\exp\{\phi^{(\lambda)}(\zeta)\}$ and defining the process $x(t)$ by $dx(t) = -\lambda x(t)dt + dz^{(\lambda)}(t)$ we have that a stationary version of $x(t)$ exists, with one-dimensional marginal distribution given by the characteristic function $c(\zeta)$. □

In the setting of Theorem 2.2, since $z^{(\lambda)}(t)$ is a homogeneous Lévy process and $\phi^{(\lambda)}(\zeta) = \lambda\phi^{(1)}(\zeta)$ we have in fact that the process $z^{(\lambda)}(t)$ is identical in law to the process $z^{(1)}(\lambda t)$. Thus, abbreviating $z^{(1)}(t)$ to $z(t)$ we have

(2.16) $$\{z^{(\lambda)}(t)\}_{t\geq 0} \stackrel{d}{=} \{z(\lambda t)\}_{t\geq 0} \ .$$

This implies that

$$x(t) \stackrel{d}{=} e^{-\lambda t}x(0) + e^{-\lambda t}\int_0^{\lambda t} e^s \mathrm{d}z(s)$$

with corresponding stochastic differential equation

$$\mathrm{d}x(t) = -\lambda x(t)\mathrm{d}t + \mathrm{d}z(\lambda t) \ .$$

In view of the relation (2.16) we will henceforth, with a slight abuse of terminology, refer to $z(t)$ (rather than $z^{(\lambda)}(t)$) as the *background driving Lévy process* (BDLP) for the Ornstein - Uhlenbeck process $x(t)$, whatever the value of the regression parameter λ.

An application of Fubini's theorem for stochastic integrals (cf. for instance Protter 1992; p. 159) shows that the cumulative process

$$s(t) = \int_0^t x(s)\mathrm{d}s$$

may be represented as

(2.17) $$s(t) = \lambda^{-1}\{z^{(\lambda)}(t) - x(t) + x(0)\} \ .$$

Specifically,

$$\begin{aligned} s(t) &= \int_0^t x(s)\mathrm{d}s \\ &= x(0)\int_0^t e^{-\lambda u}\mathrm{d}u + \int_0^t e^{-\lambda u}\mathrm{d}u \int_0^u e^{\lambda s}\mathrm{d}z^{(\lambda)}(s) \\ &= \lambda^{-1}(1 - e^{-\lambda t})x(0) + \int_0^t e^{\lambda s}\mathrm{d}z^{(\lambda)}(s)\int_u^t e^{-\lambda u}\mathrm{d}u \\ &= \lambda^{-1}\left\{(1 - e^{-\lambda t})x(0) + z^{(\lambda)}(t) - \int_0^t e^{-\lambda(t-s)}\mathrm{d}z^{(\lambda)}(s)\right\} \\ &= \lambda^{-1}\{z^{(\lambda)}(t) - x(t) + x(0)\} \ . \end{aligned}$$

Furthermore, in view of (2.16) we have

$$\{s(t)\}_{t\geq 0} \overset{d}{=} \{\lambda^{-1}(z(\lambda t) - x(t) + x(0))\}_{t\geq 0} \ . \tag{2.18}$$

The stationary process $\{x(t)\}_{t\geq 0}$ can be extended to a stationary process on the whole real line. To do this we introduce an independent copy of the process $z^{(\lambda)}(t)$ but modify it to be caglad, thus obtaining a process $\overline{z}^{(\lambda)}(t)$. Now, for $t < 0$ define $z^{(\lambda)}(t)$ and $x(t)$ by $z^{(\lambda)}(t) = \overline{z}^{(\lambda)}(-t)$ and

$$x(t) = e^{-\lambda|t|}x(0) + e^{-\lambda|t|}\int_t^0 e^{\lambda|s|}\mathrm{d}z^{(\lambda)}(s) \ .$$

Then $\{z^{(\lambda)}(t)\}_{t\in\mathbf{R}}$ is a homogeneous, cadlag Lévy process and $\{x(t)\}_{t\in\mathbf{R}}$ is a strictly stationary process of Ornstein-Uhlenbeck type.

2.5 Long range dependence, selfsimilarity and selfdecomposability

A stationary process $x(t)$ is said to be *long range dependent* if its correlation function r is asymptotically, as $\tau \to \infty$, of the form

$$r(\tau) \sim L(\tau)\tau^{-2(1-H)} \tag{2.19}$$

for some constant H with $\frac{1}{2} < H < 1$ and some slowly varying function L.

Closely related to long range dependence is the concept(s) of selfsimilarity. To indicate the relation, suppose for a moment that x is a discrete time process with correlation function satisfying (2.19). Then for every $m = 1, 2, \ldots$ the derived sum process

$$x_m(t) = x(t+1) + \ldots + x(t+m)$$

has, exactly or approximately for large τ, the same correlation function as x itself, and in this sense the processes $x^{(m)}$ are exactly or approximately selfsimilar. The correlation functions are exactly equal if

$$r(\tau) = \frac{1}{2}\left\{(\tau+1)^{2H} - 2\tau^{2H} + (\tau-1)^{2H}\right\} \ . \tag{2.20}$$

The heuristic idea is that the cumulative sum process $s(t) = x(1) + \ldots + x(t)$ will exhibit the same correlation structure whether looked at ‘close up’ or ‘from a larger distance’.

If (2.20) holds then, writing σ^2 for the variance of $x(t)$, we have that the variance of $s(t)$ satisfies

$$var\{s(t)\} = \sigma^2 t^{2H} \tag{2.21}$$

exactly, for any $H \in (0,1)$. In general, (2.21) is valid asymptotically.

Any continuous time process $s(t)$ is said to be *exactly selfsimilar* with exponent H if for any positive c the process $s_c(t) = s(ct)$ follows the same probabilistic law as the process $c^H s(t)$. Of most interest, both theoretically and practically, among such processes are those for which the increments are stationary, and examples of these are generally difficult to come by, see for instance Samorodnitsky and Taqqu (1994) and references given there. Dropping the requirement of stationary increments opens a plethora of possibilities. In particular, to any homogeneous Lévy process $z(t)$ with $z(t)$ selfdecomposable for all t and to any $H > 0$ there is associated a, uniquely determined, exactly selfsimilar process $z^H(t)$ which is cadlag with independent increments and such that $z(t)$ and $z^H(t)$ have the same distribution at time $t = 1$ (Sato 1991). Furthermore, for every $t \in \mathbf{R}_+$ the law of $z^H(t)$ is selfdecomposable. In fact, another result of Sato (1991) says that if $y(t)$, $t \geq 0$, is a stochastically continuous selfsimilar process with independent increments and $y(0) = 0$ a.s. then the distribution of $y(t)$ is selfdecomposable for every t.

2.6 Processes of type G

Processes $\{y(\eta) : \eta \in \mathscr{H}\}$ of type G were defined and studied in Rosinski (1991). Here $\mathscr{H}$ is an arbitrary index set. The most basic of these processes are of the form

$$y(\eta) = \int_0^1 f(\eta, s)\mathrm{d}z(s) \tag{2.22}$$

where $z(s)$ is a homogeneous Lévy process having the property that $z(1)$ may be represented as $z(1) = \sigma\epsilon$ where $\sigma > 0$ and ϵ are independent random variables with σ^2 infinitely divisible and ϵ being standard normal. Random variables having such a representation $\sigma\epsilon$ are said to be of *type* G. The mixture representation of the NIG distributions mentioned in Subsect. 2.1 means, in particular, that $NIG(\alpha, 0, 0, \delta)$ is of type G for any values of α and δ. For simplicity we assume henceforth that $\sup\{x : P(\sigma \geq x) = 1\} = 0$. Further, we define $\{\sigma^2(s) : s \in [0, 1]\}$ as the homogeneous Lévy process for which $\sigma^2(1) \stackrel{d}{=} \sigma^2$.

By Marcus (1987; Lemma 2.2) and Rosinski (1991), for any such type G relation (2.22) the processes $\sigma^2(s)$, $z(s)$ and $y(\eta)$ are representable in law as

$$\{\sigma^2(s) : s \in [0, 1]\} \stackrel{d}{=} \{\sum_{i=1}^{\infty} R^2(T_i)1_{[0,s]}(r_i) : s \in [0, 1]\} \ , \tag{2.23}$$

$$\{z(s) : s \in [0, 1]\} \stackrel{d}{=} \{\sum_{i=1}^{\infty} w_i R(T_i)1_{[0,s]}(r_i) : s \in [0, 1]\} \tag{2.24}$$

and

$$\{y(\eta) : \eta \in \mathscr{H}\} \stackrel{d}{=} \{\sum_{i=1}^{\infty} w_i R(T_i) f(\eta, r_i) : \eta \in \mathscr{H}\} \ . \tag{2.25}$$

In these expressions, $\{w_i\}$, $\{T_i\}$ and $\{r_i\}$ are three independent sequences of independent random variables with the w_i standard normal, the r_i uniform on $[0, 1]$

and $T_1 < T_2 < \ldots < T_i < \ldots$ the sequence of arrival times of a (independent) Poisson process of unit rate. Furthermore, the function R is defined in terms of the Lévy measure, H say, of σ^2 by $R(t) > 0$ and

$$R^2(t) = \inf\{x > 0 : H((x, \infty)) \le t\} \tag{2.26}$$

for all $t > 0$. All three series, in (2.23), (2.24) and (2.25), converge a.s.

3 Ornstein-Uhlenbeck processes of NIG and IG type

The $NIG(\alpha, \beta, \mu, \delta)$ distribution is selfdecomposable. This was shown by Halgreen (1979) and may also be seen from the expression (2.9) for the Lévy density of $NIG(\alpha, \beta, \mu, \delta)$ using Lemma 2.1 and the standard formulae for the Bessel functions K_ν

$$\begin{aligned} K_\nu(x) &= K_{-\nu}(x) \\ K'_\nu(x) &= -K_{\nu-1}(x) - \nu x^{-1} K_\nu(x) \\ K_{\nu+1}(x) &= 2\nu x^{-1} K_\nu(x) + K_{\nu-1}(x). \end{aligned} \tag{3.1}$$

It follows then from the results in Subsect. 2.4, in particular Theorem 2.3, that there exists a stationary Ornstein-Uhlenbeck process $\{x(t)\}_{t \in \mathbf{R}}$ such that $x(t) \sim NIG(\alpha, \beta, \mu, \delta)$ for every $t \in \mathbf{R}$, whatever the value of the regression parameter λ. We shall refer to this process as the NIG Ornstein-Uhlenbeck process. To study the character of the process we assume, for simplicity that $\mu = 0$ and, since $x(t) \sim NIG(\alpha, \beta, 0, \delta)$ implies $-x(t) \sim NIG(\alpha, -\beta, 0, \delta)$, we further restrict attention to the case $\beta \ge 0$.

Similarly, we shall consider the character of the stationary inverse Gaussian Ornstein-Uhlenbeck process, the existence of which is also guaranteed by Theorem 2.3.

3.1 BDLP of the NIG O-U and IG O-U processes

We proceed to derive the Lévy measure of the BDLP $\{z_t\}_{t \in \mathbf{R}}$ corresponding to the NIG Ornstein-Uhlenbeck process, using the relation (2.12). From the formulae (2.9) and (3.1) we find

(3.2)
$$\begin{aligned} w(x) =& -u(x) \\ &+ \pi^{-1}\delta\alpha \left\{ |x|^{-1} K_1(\alpha|x|) - \alpha \operatorname{sign}(x) K_1'(\alpha|x|) - \beta \operatorname{sign}(x) K_1(\alpha|x|) \right\} e^{\beta x} \\ =& \pi^{-1}\delta\alpha \left[\left\{ |x|^{-1} - \beta \operatorname{sign}(x) \right\} K_1(\alpha|x|) + \alpha K_0(\alpha|x|) \right] e^{\beta x} \\ =& (1 - \beta x) u(x) + \pi^{-1} \delta \alpha^2 K_0(\alpha|x|) e^{\beta x} \ . \end{aligned}$$

It is illuminating to rewrite this as

(3.3) $$\begin{aligned} w(x) =& \pi^{-1}\delta\alpha^2[(1-\rho)\{\alpha\,|x|^{-1} K_1\left(\alpha\,|x|\right) \\ & + K_0\left(\alpha\,|x|\right)\} + \rho D(\alpha\,|x|)]e^{\rho\alpha x} \\ =& (1-\rho)\{\bar{u}(\alpha x) + \bar{u}_0(\alpha x)\} + \rho\bar{u}_1(\alpha x) \end{aligned}$$

where $\rho = \beta/\alpha \geq 0$ and

(3.4) $$\begin{aligned} \bar{u}(x) &= \pi^{-1}\delta\alpha^2\,|x|^{-1} K_1(|x|)e^{\rho x} \\ \bar{u}_0(x) &= \pi^{-1}\delta\alpha^2 K_0(|x|)e^{\rho x} \\ \bar{u}_1(x) &= \pi^{-1}\delta\alpha^2 D(x)e^{\rho x} \end{aligned}$$

the function $D(x)$ being defined by

(3.5) $$D(x) = \{|x|^{-1} - \mathrm{sign}(x)\}K_1(|x|) + K_0(|x|)\ .$$

Using the relations (3.1) one finds that $D(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ and since $D(x) \to 0$ for $x \to \pm\infty$ we have $D(x) > 0$. Note also that $\bar{u}(\alpha x) = u(x)$.

It follows that the BDLP of the NIG Ornstein–Uhlenbeck process is a sum of three independent homogeneous Lévy processes, i.e.

(3.6) $$z(t) = y(t) + p(t) + q(t)$$

where the Lévy densities corresponding to $y(t)$, $p(t)$ and $q(t)$ at the time $t = 1$ are, respectively, $(1-\rho)\bar{u}(\alpha x)$, $(1-\rho)\bar{u}_0(\alpha x)$, and $\rho\bar{u}_1(\alpha x)$. The last process is of course degenerate in case $\beta = 0$.

From the above formulas we have immediately that $y(t)$ is the NIG Lévy process such that $y(1) \sim NIG(\alpha, \beta, 0, (1-\rho)\delta)$.

In contrast to $y(t)$, neither $p(t)$ nor $q(t)$ is selfdecomposable, as may be seen from Lemma 2.1 and the formulae (3.1).

To determine the character of the process $p(t)$ precisely we shall invoke Corollary 2.1. From Gradshteyn and Ryzhik (1965; formula 6.611.9) we have for $0 \leq |s| < \gamma$

(3.7) $$\int_0^\infty e^{-sx} K_0(\gamma x)dx = (\gamma^2 - s^2)^{-1/2} \arg\cos(s/\gamma)\ .$$

Thus, by Corollary 2.1, the cumulant generating function of $p(1)$ is given by $k_0(\theta) - k_0(0)$ where

(3.8) $$\begin{aligned} k_0(\theta) =& (1-\rho)\pi^{-1}\delta\alpha^2 \int_{\mathbf{R}\setminus\{0\}} e^{\theta x} K_0(\alpha|x|)e^{\beta x}dx \\ =& (1-\rho)\delta\alpha^2\{\alpha^2 - (\theta+\beta)^2\}^{-1/2} \end{aligned}$$

for $0 \leq |\theta + \beta| < \alpha$.

Now, consider the Laplace transform of $p(1)$, i.e. $L_0(\theta) = \exp\{k_0(\theta) - k_0(0)\}$ and let

$$\xi = \{(1-\rho)/(1+\rho)\}^{1/2}\delta\alpha \ .$$

By Taylor expansion we find

$$\begin{aligned} L_0(\theta) \quad &= \quad e^{-\xi}\sum_{\nu=0}^{\infty}\frac{\xi^\nu}{\nu!}[\{1-(\theta+\beta)/\alpha\}/(1-\rho)^{1/2}]^{-\nu/2} \\ &\qquad \times[\{1+(\theta+\beta)/\alpha\}/(1+\rho)^{1/2}]^{-\nu/2} \end{aligned} \tag{3.9}$$

which shows that $p(1)$ is of the form

$$p(1) = \frac{1}{2}\alpha^{-1}(1-\rho^2)^{-1/2}\sum_{i=1}^{N}(u_i^2 - u_i'^2) \tag{3.10}$$

with N denoting a Poisson variate with mean ξ and the u_i and u_i' being independent standard normally distributed and independent of N.

An equally explicit representation is not available for the process $q(t)$. To study $q(t)$ we invoke Corollary 2.2 according to which $q(1)$ has a cumulant generating function of the form $g(\theta) - g(0)$ where $g(\theta)$ satisfies

$$g''(\theta) = \rho\pi^{-1}\delta\alpha^2\int_{\mathbf{R}\setminus\{0\}} e^{\theta x}x^2 D(\alpha x)e^{\beta x}\,\mathrm{d}x \ .$$

In view of (3.5), together with (3.4) and (3.8), we have that $g(\theta)$ is the sum of three terms

$$g(\theta) = h(\theta) + h_0(\theta) + h_1(\theta)$$

where

$$h(\theta) = \rho\delta\{\alpha^2 - (\theta+\beta)^2\}^{1/2}$$

$$h_1(\theta) = \rho\delta\alpha^2\{\alpha^2 - (\theta+\beta)^2\}^{-1/2}$$

and

$$\begin{aligned} h_0''(\theta) =& \rho\pi^{-1}\delta\alpha^2\left\{\int_{-\infty}^{0} e^{\theta x}x^2K_1(\alpha\,|x|)e^{\beta x}\mathrm{d}x - \int_0^\infty e^{\theta x}x^2K_1(\alpha x)e^{\beta x}\mathrm{d}x\right\} \\ =& -2\rho\pi^{-1}\delta\alpha^2\int_0^\infty \sinh\{(\theta+\beta)x\}x^2K_1(\alpha x)\mathrm{d}x \ . \end{aligned}$$

Now, since $xK_1(x)$ is integrable near 0 ($K_1(x) \sim x^{-1}$ as $x \downarrow 0$), we must have

$$h_0'(\theta) = -2\rho\pi^{-1}\delta\alpha^2\int_0^\infty \cosh\{(\theta+\beta)x\}xK_1(\alpha x)\mathrm{d}x \ . \tag{3.11}$$

By (3.1) we have $K_1(x) = -K_0'(x)$ and hence, by partial integration, we find

$$\int_0^\infty e^{sx} x K_1(\alpha x) \mathrm{d}x = \alpha^{-1} \int_0^\infty (sx+1) K_0(\alpha x) e^{sx} \mathrm{d}x$$
$$= \alpha^{-1} \frac{\partial}{\partial s} \left\{ s \int_0^\infty K_0(\alpha x) e^{sx} \mathrm{d}x \right\}$$

and from this it follows, by (3.7) and (3.11), that

$$h_0(\theta) = -\rho\delta\alpha(\theta+\beta)\{\alpha^2 - (\theta+\beta)^2\}^{-1/2} .$$

Collecting terms we obtain

$$\begin{aligned} g(\theta) =& \rho\delta\{\alpha^2 - (\theta+\beta)^2\}^{-1/2}[-\{\alpha^2 - (\theta+\beta)^2\} - \alpha(\theta+\beta) + \alpha^2] \\ =& -\rho\delta\{\alpha^2 - (\theta+\beta)^2\}^{-1/2}(\theta+\beta)\{\alpha - (\theta+\beta)\} \\ =& -\rho\delta(\theta+\beta)\{(\alpha-\theta-\beta)/(\alpha+\theta+\beta)\}^{1/2} . \end{aligned}$$

It does not seem possible from this to give an elementary description of the distribution of $q(1)$. In particular, the procedure used to derive (3.10) does not work in the present case.

All in all we have shown

Theorem 3.1. *The BDLP $z(t)$ for the normal inverse Gaussian Ornstein-Uhlenbeck process with parameters $(\alpha, \beta, 0, \delta)$ is, for $\beta \geq 0$, representable as the sum of three independent homogeneous Lévy processes: $z(t) = y(t) + p(t) + q(t)$. The first process $y(t)$ is a normal inverse Gaussian Lévy process, with parameters $(\alpha, \beta, 0, (1-\rho)\delta)$, and the second has the form*

$$p(t) = \frac{1}{2}\alpha^{-1}(1-\rho^2)^{-1/2} \sum_{i=1}^{N_t} (u_i^2 - u_i'^2) \tag{3.12}$$

where N_t denotes a Poisson process with rate $[\{(1-\rho)/(1+\rho)\}^{1/2}\delta\alpha]^{-1}$ and the u_i and u_i' $(i = 0, 1, 2, \ldots)$ are independent standard normally distributed and independent of the process N_t. Finally, the Laplace transform $E\exp(\theta q(t))$ of $q(t)$ is

$$\begin{aligned} &\exp\Big(t\rho\delta\Big[\beta\{(\alpha-\beta)/(\alpha+\beta)\}^{1/2} \\ &\quad -(\theta+\beta)\{(\alpha-\theta-\beta)/(\alpha+\theta+\beta)\}^{1/2}\Big]\Big) . \end{aligned} \tag{3.13}$$

□

The same kind of analysis as that given above for the normal inverse Gaussian Ornstein-Uhlenbeck process can be applied to the $IG(\delta, \gamma)$ distribution. The Lévy density $IG(\delta, \gamma)$ is

$$u(x) = (2\pi)^{-1/2}\delta x^{-3/2} e^{-\gamma^2 x/2} ,$$

and it follows that $w(x)$, as given by (2.12), takes the form

(3.14) $$w(x) = (2\pi)^{-1/2}\frac{\delta}{2}\left\{x^{-1}+\gamma^2\right\}\ x^{-1/2}e^{-\gamma^2 x/2}.$$

This implies that the process $z(t)$ driving the inverse Gaussian Ornstein-Uhlenbeck process (i.e. the stationary Ornstein-Uhlenbeck type process having $IG(\delta,\gamma)$-distributed one-dimensional marginals) is a sum of two independent processes, $z(t) = y(t)+p(t)$, where $y(t)$ is an inverse Gaussian Lévy process with parameters $\delta/2$ and γ for $y(1)$, while $p(t)$ is of the form

(3.15) $$p(t) = \gamma^{-2}\sum_{i=1}^{N_t} u_i^2$$

with N_t a Poisson process of rate $\{\delta\gamma/2\}^{-1}$ and the u_i being independent standard normal and independent of the process N_t.

3.2 Likelihood analysis

We now consider some questions concerning likelihood analysis for discretely observed stationary NIG and IG processes of Ornstein-Uhlenbeck type. Suppose $x(t)$ is such a process and that it has been observed at the times $t = 1, \ldots, n$.

First, let $x(t)$ be the NIG Ornstein-Uhlenbeck process. In a financial context it will often be reasonable, at least initially, to assume that the parameters μ and β are both 0 and we do this here. An explicit expression for the likelihood function of the parameters α, δ and λ, where λ is the regression parameter, cf. formula (2.13), is not available. However, the likelihood function can be accurately calculated, as will now be discussed.

Since $x(t)$ satisfies $x(t+1) = e^{-\lambda t}x(t) + \overline{z}(t)$, where

$$\overline{z}(t) = e^{-\lambda-\lambda t}\int_t^{t+1} e^{\lambda s}\mathrm{d}z(\lambda s)$$

and the BDLP $z(t)$ is described in Theorem 3.1, the problem lies in determining the probability density function of $\overline{z}(t)$ at least up to a constant not depending on (α,δ,λ), and since $z(t)$ has stationary increments we need just consider calculation of the probability density function of $\overline{z}(0)$ or, equivalently, of $e^{\lambda}\overline{z}(0)$.

Now, let $z^*(t)$, $t \in [0,1]$, be the process

$$z^*(t) = \int_0^t e^{\lambda s}\mathrm{d}z(\lambda s)\ .$$

Then $z^*(1) = e^{\lambda}\overline{z}(0)$ and $z^*(t)$ is Markovian, in fact a process with independent increments. The density function of $z^*(1)$ at an arbitrary point x can therefore be calculated as

(3.16) $$p(x;\alpha,\delta,\lambda) = E\left\{p(x;\alpha,\delta,\lambda \mid z^*(\tau))\right\}$$

the expectation being over the values of the process $z^*(\cdot)$ at an, arbitrarily chosen, time τ. If τ is taken close to 1 we have approximately, letting $\varepsilon = 1 - \tau$,

$$p(x;\alpha,\delta,\lambda \mid z^*(\tau)) \doteq \int_{-\infty}^{\infty} \nu(x - z^*(\tau) - y; e^{\lambda}\alpha, z^*(\tau), e^{-\lambda}\lambda\varepsilon\delta)\Xi_{\varepsilon}(\mathrm{d}y;\alpha,\delta,\lambda). \tag{3.17}$$

Here $\nu(\cdot;\alpha,\mu,\delta)$ denotes the density function of the $NIG(\alpha,0,\mu,\delta)$ law and $\Xi_{\varepsilon}(\cdot;\alpha,\delta,\lambda)$ is the distribution function of a random variable of the form

$$e^{\lambda}\alpha^{-1}\sum_{i=1}^{N}(u_i^2 - u_i'^2)/2$$

where N is a Poisson variate of mean $\lambda\varepsilon\delta\alpha$ and the u_i and u_i' are independent standard normal and independent of N (cf. Theorem 3.1). Numerical determination of the integral in formula (3.17) is quite feasible, especially for small values of ε. In this connection the following two distributional results are helpful:

$$(u_1^2 - u_1'^2)/2 \sim \pi^{-1}K_0(|x|) \tag{3.18}$$

$$(u_1^2 + u_2^2 - u_1'^2 - u_2'^2)/2 \sim \frac{1}{2}e^{-|x|} \tag{3.19}$$

where the right hand sides are the probability densities of the random variables on the left. Formula (3.18) follows from formula 3.364.3 in Gradshteyn and Ryzhik (1965). Furthermore, for $n \to \infty$ the distribution of $\Sigma_1^n(u_i^2 - u_i'^2)/2$ rapidly approaches normality.

The remaining problem is then the calculation of the expectation in (3.16) which can be carried out approximately by approximate simulation of $z^*(\tau)$ as follows.

Let n be a large integer and let the processes $y(t)$ and $p(t)$ be as in Theorem 3.1 (with $\mu = \beta = 0$). The value of the process $z^*(\cdot)$ at time τ is approximately given by

$$z^*(\tau) \doteq \sum_{i=1}^{n} e^{\lambda i/n}(g_i + h_i) \tag{3.20}$$

where $g_i = y(\lambda\tau i/n) - y(\lambda\tau(i-1)/n)$ and $h_i = p(\lambda\tau i/n) - p(\lambda\tau(i-1)/n)$. By the characterization of y_t and p_t given in Theorem 3.1, the random variables g_i are i.i.d. normal inverse Gaussian and the h_i are i.i.d. and compound Poisson of the form

$$\frac{1}{2}\alpha^{-1}\sum_{i=1}^{N_{\lambda\tau/n}}(u_i^2 - u_i'^2)\ . \tag{3.21}$$

To simulate an NIG random variable g one may use the representation $g = \sigma\epsilon$ where σ and ϵ are independent with σ^2 IG-distributed and ϵ standard normal. A simple procedure for simulating IG variates has been given by Michael et al. (1976); this is based on simulation of standard normal variates. Simulation of (3.21) can also be carried out simply from independent standard normal variables.

An alternative, and in some ways preferable, approach to approximate simulation of $z^*(\tau)$ is to use the series representation of processes of type G. By Theorem 3.1 we have, since β has been assumed to be 0, that

$$z^*(\tau) = \int_0^\tau e^{\lambda s} \mathrm{d}y(s) + \frac{1}{2}\alpha \sum_{i=1}^{N_{\lambda\tau}} \exp(T_i')(u_i^2 - u_i'^2) \tag{3.22}$$

where $T_1' < T_2' < \ldots < T_i' < \ldots$ are the arrival times of the Poisson process N_t whose rate is $\delta\alpha$. Furthermore, $y(t)$ is the $NIG(\alpha, 0, 0, \lambda\delta)$ Lévy process and, as noted in Subsect. 2.6, $y(t)$ is of type G. Hence, using (2.25), we find that the integral in (3.22) satisfies

$$\int_0^\tau e^{\lambda s} \mathrm{d}y(\lambda s) \sim \sum_{i=1}^{\infty} w_i R(T_i) \exp(\lambda r_i) I_{[0,\tau]}(r_i) \tag{3.23}$$

with $\{w_i\}$, $\{T_i\}$ and $\{r_i\}$ as in formula (2.25) and the function R determined by (2.26) with H being the Lévy measure of the inverse Gaussian law $IG(\lambda\delta, \alpha)$. The series on the right hand side of (3.23) converges very rapidly.

To determine the likelihood function the approximate simulation of $z^*(\tau)$ based on (3.20) or on (3.22-3) has to be carried out for a range of values of $(\alpha, \delta, \lambda)$. However, as follows from the above, one and the same set of $N(0, 1)$ pseudo-variates can be used in all cases, and this reduces the computation time considerably. Furthermore, in selecting a suitable range of values of $(\alpha, \delta, \lambda)$ it is requisite to have a good initial estimate of $(\alpha, \delta, \lambda)$; for most purposes it should suffice to estimate α and δ by treating the data as if they constituted an i.i.d. sample from a $NIG(\alpha, 0, 0, \delta)$ distribution and estimating λ from the empirical correlation coefficient.

The technique outlined here applies equally to the stationary IG Ornstein-Uhlenbeck process, using results for that process given above.

It should be noted that the technique for likelihood calculation discussed here is partly similar to a method introduced by Pedersen (1995a) for maximum likelihood estimation under discretely observed stochastic processes determined by stochastic differential equations; see also Pedersen (1995b).

4 (Quasi) long range dependent NIG processes

As indicated in the Introduction, series of logarithmic asset returns generally exhibit correlation patterns in the character of moderate to long range dependence, a feature that we shall refer to as *(quasi) long range dependence* or QLRD, for short. As a possible approach to modelling this feature as well as the typical distributional behaviour, in this section we construct classes of stationary QLRD processes whose one-dimensional marginal distributions are either normal inverse Gaussian or inverse Gaussian. The NIG and IG Ornstein-Uhlenbeck processes discussed in Sect. 3 may be used as building blocks for such QLRD processes.

Let $x^{(k)}(\cdot)$, $k = 1, 2, \dots$, be a sequence of independent and stationary processes such that for all k and $t \in \mathbf{R}$ the distribution of $x^{(k)}(t)$ is normal inverse Gaussian with parameters $(\alpha, \beta, 0, \delta_k)$. For instance, $x^{(k)}(\cdot)$ could be of Ornstein-Uhlenbeck type.

Theorem 4.1. *Suppose that δ_k satisfies*

$$\delta_k \sim const. \cdot k^{-1-2(1-H)} \tag{4.1}$$

for $k \to \infty$ and some $H \in (0, 1)$, and let $\delta = \Sigma_{k=1}^{\infty} \delta_k$.

The process

$$x(t) = \sum_{k=1}^{\infty} x^{(k)}(k^{-1}t) \tag{4.2}$$

is stationary and welldefined as an L^2 limit, and the marginal distribution of $x(t)$ is $NIG(\alpha, \beta, 0, \delta)$.

Furthermore, if the processes $x^{(k)}(\cdot)$ all have the same correlation function $r(\cdot)$ and if $r(\cdot)$ is continuous and $r(\tau) \to 0$ for $\tau \to \infty$ then the correlation function $\bar{r}$ of $x(\cdot)$ satisfies

$$\bar{r}(\tau) \sim L(\tau)\tau^{-2(1-H)} \tag{4.3}$$

for some slowly varying function L. Thus, if $\frac{1}{2} < H < 1$ the process exhibits long range dependence with exponent H. □

Proof. Since the variance of $x^{(k)}(k^{-1}t)$ is of the form

$$var\{x^{(k)}(k^{-1}t)\} = \{\alpha(1 - \rho^2)^{3/2}\}^{-1}\delta_k$$

we have that $x(\cdot)$ is welldefined in the sense of L^2 convergence. That $x(t)$ is $NIG(\alpha, \beta, 0, \delta)$ then follows from the convolution property of the normal inverse Gaussian distribution.

The correlation function of $x(t)$ is of the form

$$\bar{r}(\tau) = \delta^{-1} \sum_{k=1}^{\infty} \delta_k r(k^{-1}\tau)$$

from which the asymptotic behaviour (4.3) may be derived by a simple calculation. □

The same conclusions as reached above for NIG processes will hold for an IG process of the form (4.2) with the $x^{(k)}(t)$ as independent and stationary processes such that $x^{(k)}(t) \sim IG(\delta_k, \alpha)$, in which case we have $x(t) \sim IG(\delta, \alpha)$.

Ordinary likelihood analysis of these QLRD processes does not seem feasible, but the Whittle procedure, which is based on the smoothed periodogram, is likely to open the way for a quasi-likelihood approach. See Heyde (1997) for a discussion of the Whittle procedure in the context of estimating functions.

A comprehensive empirical study, Guillaume et al. (1994) (see also Müller et al. 1990, 1993; Schnidrig and Würtz 1994), indicates that for free floating currencies on the foreign exchange market the logarithmic price changes closely follow a scaling law with exponent $H = 0.58$. More specifically, thinking for instance of the USDDEM exchange rate as this has developed over a period $[0, T]$, let $q_i(\Delta t)$ (where $\Delta t > 0$) denote the change in logarithmic price over the time interval $[(i-1)\Delta t, i\ \Delta t]$, $i = 1, 2, \ldots$. Then, letting $n = T/\Delta t$ (which, for simplicity, we assume to be an integer), the investigations referred to above show that, over several orders of magnitude in time, the cumulative sums of the absolute values

$$Q_n(\Delta t) = |\ q_1(\Delta t)\ | + |\ q_2(\Delta t)\ | + \ldots + |\ q_n(\Delta t)\ | \tag{4.4}$$

very nearly satisfy a linear relationship of the form

$$\log E\{Q_n(\Delta t)\} = H \log \Delta t + const.\ . \tag{4.5}$$

Assuming that this reflects a full scaling law, in the sense that $\{q_1(t)\}_{t \in \mathbf{R}_+}$ is a selfsimilar process with exponent H, it seems of some interest to consider modelling of observed sequences of the form $q_1(\Delta t), q_2(\Delta t), \ldots, q_n(\Delta t)$ by theoretical sequences $s(\Delta t), s(2\Delta t) - s(\Delta t), \ldots, s(n\Delta t) - s((n-1)\Delta t)$ where

$$s(t) = \int_0^t x(s)\mathrm{d}s$$

and where $x(t)$ is a stationary process of the form discussed in Theorem 4.1 with $H = 0.58$ (cf. formula (2.21)). A discussion of this will be given elsewhere.

5 A selfsimilar NIG process with independent increments

In connection with the results on selfsimilarity and selfdecomposability mentioned in Subsect. 2.5, it seems worth noting that the exactly selfsimilar process $z^H(t)$ associated, in the sense of Sato (1991), to the NIG Lévy process $z(t)$ with parameters $(\alpha, \beta, \mu, \delta)$ also has the property that its marginal distributions are normal inverse Gaussian. More specifically, while

$$z(t) \sim NIG(\alpha, \beta, t\mu, t\delta)$$

it can be shown that, in terms of the invariant parameters $\bar{\alpha}$ and $\bar{\beta}$, the process $z^H(\cdot)$ satisfies

$$z^H(t) \sim NIG[\bar{\alpha}, \bar{\beta}, t^H\mu, t^H\delta] \ .$$

In fact, by Proposition 4.1 in Sato (1991) the Lévy measure υ_t^H of $z^H(t)$ is related to the Lévy measure υ_t of $z(t)$ by the formula

$$\upsilon_t^H(B) = \int 1_B(t^H x)\upsilon_1(\mathrm{d}x) \tag{5.1}$$

holding for all Borel sets B. If υ_t has a density, u_t say, with respect to Lebesgue measure then so has υ_t^H and then (5.1) may be recast as

$$u_t^H(x) = t^{-H}u_1(t^{-H}x)$$

with u_t^H denoting the density of υ_t^H. Specializing to the case where $z(t)$ is the NIG Lévy process with parameters $(\alpha, \beta, \mu, \delta)$ we have $u_1(x) = \pi^{-1}\delta\alpha\,|x|^{-1}K_1(\alpha\,|x|)$ $\times e^{\beta x}$ (cf. formula (2.9)) and hence

$$u_t^H(x) = \pi^{-1}\delta\alpha\,|x|^{-1}K_1(\alpha\left|t^{-H}x\right|)\exp(\beta t^{-H}x) \ .$$

This is the Lévy density of $NIG(t^{-H}\alpha, t^{-H}\beta, t^H\mu, t^H\delta) = NIG[\bar{\alpha}, \bar{\beta}, t^H\mu, t^H\delta]$.

It is an open question whether there exist nontrivial selfsimilar processes with stationary normal inverse Gaussian increments.

6 Direct superposition of Ornstein-Uhlenbeck NIG and IG processes

Superposition of Ornstein-Uhlenbeck or autoregressive processes offers one approach to parsimonious modelling of marginal laws and long-range-like dependence. In the context of turbulence this was discussed in Barndorff-Nielsen et al. (1990, 1993), see also Barndorff-Nielsen et al. (1995). Here we shall briefly consider such superpositions in relation to the normal inverse Gaussian and the inverse Gaussian laws, partly as a preparation for the following section. In Sect. 4 we also considered superpositions of stationary NIG processes, in particular processes of the Ornstein-Uhlenbeck type, but the character of those superpositions is different from that discussed below.

Let $x^{(i)}$, $i = 1, \ldots, m$, be independent and stationary NIG Ornstein-Uhlenbeck processes with regression parameters λ_i, $i = 1, \ldots, m$, and let $x = x^{(1)} + \ldots + x^{(m)}$. Assuming that the parameters of the normal inverse Gaussian distribution of $x^{(i)}(t)$ are $(\alpha, \beta, \mu_i, \delta_i)$ we have that x is stationary and that $x(t) \sim NIG(\alpha, \beta, \mu, \delta)$ where $\mu = \mu_1 + \ldots + \mu_m$ and $\delta = \delta_1 + \ldots + \delta_m$.

In the context of financial time series it will often be reasonable to take $\mu_1 = \ldots = \mu_m = 0$ and $\beta_1 = \ldots = \beta_m = 0$ and we assume henceforth that this is the situation. Further, for simplicity, we restrict attention to the case $m = 2$. Thus $x = x^{(1)} + x^{(2)} \sim NIG(\alpha, 0, 0, \delta)$ and the correlation of $x(s)$ and $x(t)$, for $s \leq t$ is

$$e^{-(t-s)\lambda_1}\bar{\delta}_1 + e^{-(t-s)\lambda_2}\bar{\delta}_2 \tag{6.1}$$

where $\bar{\delta}_i = \delta_i/\delta$.

Similarly, if $x^{(1)}$ and $x^{(2)}$ are independent and stationary IG Ornstein-Uhlenbeck processes with parameters $(\delta_1, \gamma, \lambda_1)$ and $(\delta_2, \gamma, \lambda_2)$, respectively, and if $x = x^{(1)} + x^{(2)}$ and $\delta = \delta_1 + \delta_2$ then x is a stationary process having $IG(\delta, \gamma)$-distributed one-dimensional marginals and correlation function (6.1).

The likelihood function for a model of this kind is not explicitly available. However, by viewing the model in a state space framework the likelihood becomes accessible by Markov Chain Monte Carlo procedures of the type discussed by Geyer (1997) and Shephard and Pitt (1997). A more detailed discussion will be given elsewhere.

It may, incidentally, be noted that questions of moduli of continuity and large increments of infinite sums of Gaussian Ornstein-Uhlenbeck processes have been studied recently in papers by Csáki et al. (1991) and Lin (1995). See also Walsh (1981).

7 Stochastic volatility models of NIG type

We shall discuss models in one or more dimensions, first for discrete time and then in continuous time in a subordination setting.

7.1 Discrete time

Consider discrete time stationary processes of the form

$$x_t = \sigma_t \epsilon_t \tag{7.1}$$

where σ_t is positive, $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ are independent standard normal and the processes σ_t and ϵ_t are independent.

A review of work on processes of this and closely related types is given in Shephard (1996). The key approach discussed by Shephard consists in forming

$$\log x_t^2 = \log \sigma_t^2 + \log \epsilon_t^2$$

and then writing $\log \sigma_t^2$ as a linear combination of some of the previous values of $\log \sigma_t^2$ plus a normal error term. Shephard refers to this as the *log-normal stochastic volatility model*.

The processes to be discussed below are of a different character, both in terms of the marginal laws and the dependence structure.

The autocorrelation function of the process (7.1) (σ_t and ϵ_t being independent) is identically 0. However, unless the σ_t are mutually independent, both of the processes σ_t^2 and x_t^2 have nonvanishing autocorrelations. Their autocovariance functions are in fact identical and, denoting the autocorrelation function of σ_t^2 by r, we have

$$r(u) = corr\{\sigma_t^2 \sigma_{t+u}^2\} = q \cdot corr\{x_t^2 x_{t+u}^2\}$$

where q is the ratio of the variances of x_t^2 and σ_t^2 which may be written as

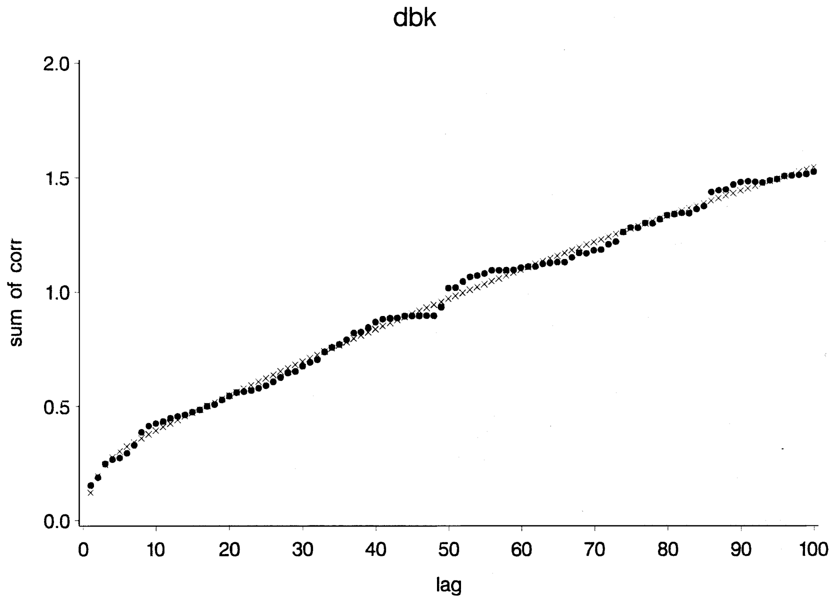

Fig. 1. Cumulative autocorrelations of squared daily log returns of Deutsche Bank stock prices from the period 2 Oct 1989–29 Dec 1995: empirical (•) and fitted theoretical (×). (Theoretical function of the form (7.2))

$$q = 3 + 2E(\sigma_t^2)^2/var\{\sigma_t^2\} \ .$$

Now, let σ_t^2 be a stationary IG process, for instance one of the IG processes considered in Sects. 4 and 6. Then the distribution of x_t is normal inverse Gaussian and quasi long range dependence may be included in the model. The full law of the process x_t is however different from that of the stationary NIG processes in Sects. 3, 4 and 6.

Suppose, in particular, that σ_t^2 is a superposition, $\sigma_t^2 = \tau_{0t} + \tau_{1t}$, of two IG Ornstein-Uhlenbeck processes, as in Sect. 6, and consider the cumulative autocorrelation function

$$r^*(m) = r(1) + \ldots + r(m)$$

which is then (cf. formula (6.1)) of the form

$$r^*(m) = w\rho_0(1-\rho_0^m)/(1-\rho_0) + (1-w)\rho_1(1-\rho_1^m)/(1-\rho_1) \tag{7.2}$$

for a $w \in [0,1]$ and $\rho_0, \rho_1 \in (0,1)$. An illustration is provided by Fig. 1 which shows the empirical cumulative autocorrelation function for the squares of the daily log returns of Deutsche Bank stock prices, calculated from the data corresponding to the period 2 October 1989–29 December 1995, the total number of log returns being 1562. The figure also shows the theoretical cumulative autocorrelation function $q^{-1}r^*(m)$ fitted to the data by a nonlinear regression procedure, the fitted parameter values being $q = 4.75$, $w = 0.921$, $\rho_0 = 0.544$ and $\rho_1 = 0.995$.

A possible interpretation is that τ_{0t} expresses an overall volatily of the market whereas τ_{1t} is specific to the particular asset considered (Deutsche Bank, in the present case). Incidentally, the present type of model is reminiscent of an extension of the log-normal stochastic volatility model indicated briefly in Subsect. 1.3.4 of Shephard (1996).

As a multivariate generalisation, let $e_t = (e_{1t}, \ldots, e_{mt})$ be an m-dimensional time series which may be thought of roughly as representing the joint price development of m stocks that would have prevailed if there were no stochastic volatility effects. More specifically, we assume that e_t is m-dimensional normal with mean 0 and a constant correlation matrix C and that the e_t are independent over time. The variances of the coordinates of e_t are however allowed to depend on t, and we denote the variance of e_{it} by σ_{it}^2. Next we allow the σ_{it}^2 to be random variables. To reflect the changed character of the price process we shall write x_t instead of e_t. We suppose that $\sigma_t^2 = (\sigma_{1t}^2, \ldots, \sigma_{mt}^2)$ is a stationary process whose one-dimensional marginals are IG-distributed. In particular, we may take σ_{it}^2, $i = 1, ..., m$, to be of the form

$$\sigma_{it}^2 = \tau_{0t} + \tau_{it}$$

where $\tau_{0t}, \tau_{1t}, \ldots, \tau_{mt}$ are independent and stationary IG processes of Ornstein-Uhlenbeck type with regression parameters $\lambda_0, \lambda_1, \ldots, \lambda_m$ and such that

$$\tau_{it} \sim NIG(\delta_i, \alpha)$$

$i = 0, 1 \ldots, m$, and then

$$\sigma_{it}^2 \sim IG(\delta_0 + \delta_i, \alpha)\ .$$

As above, one may think of τ_{0t} as expressing the volatily of the market in an overall sense whereas τ_{it} is specific to the i-th asset. The process x_t is representable in stochastic volatility form as

$$x_t = (x_{1t}, ..., x_{mt}) = (\sigma_{1t}\epsilon_{1t}, ..., \sigma_{mt}\epsilon_{mt})$$

where the $\epsilon_t = (\epsilon_{1t}, ..., \epsilon_{mt})$ are i.i.d. normal with mean 0, variance matrix C and such that the process ϵ_t is independent of the process σ_t^2.

Fitting and analysis by likelihood of the multivariate model proposed here appears feasible via state space considerations and Markov Chain Monte Carlo techniques.

An alternative type of discrete time stochastic volatility NIG models, which can be viewed as being of the form (7.1) but where the processes σ and ϵ are not independent, have been discussed in Barndorff-Nielsen (1996a). In contrast to the above models, those in the paper cited allow an explicit expression of the likelihood function; furthermore, they have some similarity to ARCH models. On the other hand, the one-dimensional marginal distributions are not normal inverse Gaussian.

7.2 Continuous time and subordination

In continuous time, modelling of stochastic volatility by subordination has considerable appeal. As an indication of the potentialities relating to NIG laws, suppose that w_t is m-dimensional Brownian motion and let $\epsilon_t = w_t A$ where A is a deterministic nonsingular $m \times m$ matrix such that $C = A^\top A$ is a correlation matrix, i.e. the diagonal elements of C are all equal to 1. Further, let $\sigma_t^2 = (\sigma_{1t}^2, ..., \sigma_{mt}^2)$ be a stationary m-dimensional process such that $\sigma_{it}^2 \sim IG(\delta_i, \alpha_i)$, $i = 1, ..., m$ and $t \in [0, \infty)$. For instance, the σ_{it}^2 may be sums $\tau_{0t} + \tau_{it}$ of independent IG Ornstein-Uhlenbeck processes, as in one of the discrete time models considered above. Now define $\zeta_t = (\zeta_{1t}, ..., \zeta_{mt})$ by

$$\zeta_{it} = \int_0^t \sigma_{is}^2 \mathrm{d}s \ .$$

The subordinated process

$$x_t = (x_{1t}, ..., x_{mt}) = (\epsilon_{\zeta_{1t}}, ..., \epsilon_{\zeta_{mt}}) \ ,$$

which in the present context we think of as representing the processes of stock prices, has uncorrelated increments. The increments of x_{it} are not NIG distributed but the law of $x_{i,t+\Delta t} - x_{it}$ will, under relatively weak conditions, be approximately NIG to a practically useful degree of accuracy.

As to the higher order correlation structure of the increments of x_t, suppose for simplicity that $m = 1$ and, for $0 < s < t < u < v$, let

$$\bar{R}(t-s, v-u; u-t) = cov\{(x_t - x_s)^2, (x_v - x_u)^2\} \ ,$$

the covariance of $(x_t - x_s)^2$ and $(x_v - x_u)^2$. Then, writing σ_t^2 for σ_{1t}^2, δ for δ_1 and α for α_1 (recall that $m = 1$) and denoting the correlation function of σ_t^2 by $r(\cdot)$ we have

$$\begin{aligned}
\bar{R}(t-s, v-u; u-t) &= E\{(\zeta_t - \zeta_s)(\zeta_v - \zeta_u)\} - E\{\zeta_t - \zeta_s\} E\{\zeta_v - \zeta_u\} \\
&= \int_s^t \int_u^v E\{\sigma_\xi^2 \sigma_\eta^2\} \mathrm{d}\eta \mathrm{d}\xi - \int_s^t E\{\sigma_\xi^2\} \mathrm{d}\xi \int_u^v E\{\sigma_\eta^2\} \mathrm{d}\eta \\
&= \int_s^t \int_u^v V\{\sigma_\xi^2, \sigma_\eta^2\} \mathrm{d}\eta \mathrm{d}\xi \\
&= (\delta/\alpha^3) \int_s^t \int_u^v r(\eta - \xi) \mathrm{d}\xi \mathrm{d}\eta \ .
\end{aligned}$$

In particular, if σ_t^2 is the IG Ornstein-Uhlenbeck process we obtain

$$\bar{R}(t-s, v-u; u-t) = (\delta/\alpha^3)\lambda^{-2}(1 - e^{-\lambda(t-s)}) e^{-\lambda(u-t)} (1 - e^{-\lambda(v-u)}) \ .$$

8 Conclusion

Taking the normal inverse Gaussian law as a building element, the present paper has explored possibilities for the construction of analytically and statistically tractable stochastic processes that have potential for capturing key stylised features of observational series from finance (and turbulence, cf. Subsect. 2.0). It is intended in future work to compare the models proposed here with a variety of data sets.

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