The Generalized Hyperbolic Model: Financial Derivatives and Risk Measures

Ernst Eberlein[1] and Karsten Prause[2]

[1] Institute for Mathematical Stochastics and Freiburg Center for Data Analysis and Modelling (FDM), University of Freiburg, Eckerstr. 1, D-79104 Freiburg, Germany

[2] HypoVereinsbank, Credit Risk Controlling, Arabellastr. 12, D-81925 München, Germany

Abstract. Statistical analysis of data from the financial markets shows that generalized hyperbolic (GH) distributions allow a more realistic description of asset returns than the classical normal distribution. GH distributions contain as subclasses hyperbolic as well as normal inverse Gaussian (NIG) distributions which have recently been proposed as basic ingredients to model price processes. GH distributions generate in a canonical way Lévy processes, i.e. processes with stationary and independent increments. We introduce a model for price processes which is driven by generalized hyperbolic Lévy motions. This GH model is a generalization of the hyperbolic model developed by Eberlein and Keller (1995). It is incomplete. We derive an option pricing formula for GH driven models using the Esscher transform as martingale measure and compare the prices with classical Black-Scholes prices. The objective of this study is to examine the consistency of our model assumptions with the empirically observed price processes for underlyings and derivatives. Finally we present a simplified approach to the estimation of high-dimensional GH distributions and their application to measure risk in financial markets.

1 Introduction

Generalized hyperbolic (GH) distributions were introduced by Ole E. Barndorf-Nielsen (1977) in the context of the sand project as a variance-mean mixture of normal and generalized inverse Gaussian (GIG) distributions.

These distributions seem to be tailor-made to describe the statistical behaviour of asset returns. Analyzing financial time series such as stock prices, indices, FX-rates or interest rates, one gets empirical distributions with a rather typical shape. They place substantial probability mass near the origin, have slim flanks and a number of observations far out in the tails. The normal distribution on which the classical models in finance are based, fails in all three aspects. How far this deviation from normality goes, depends on the time scale of the underlying data sets.

For long term studies based on weekly or even monthly data points the empirical distributions are close to the normal. But using scarce data sets effectively ignores a lot of information. Daily data is the minimum one has to consider for most purposes. Analyzing intraday data, i.e. looking at price movements on a microscopic scale leads to a deeper understanding of the relevant processes.

Definition 1. For $x \in \mathbb{R}$ the density of the *generalized hyperbolic distribution* is defined as

$$\begin{aligned} \text{gh}(x;\lambda,\alpha,\beta,\delta,\mu) &= a(\lambda,\alpha,\beta,\delta)\left(\delta^2+(x-\mu)^2\right)^{(\lambda-1/2)/2} \\ &\quad\times K_{\lambda-1/2}\big(\alpha\sqrt{\delta^2+(x-\mu)^2}\,\big)\exp\big(\beta(x-\mu)\big) \\ a(\lambda,\alpha,\beta,\delta) &= \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\,\alpha^{\lambda-1/2}\,\delta^\lambda K_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\,\big)}, \end{aligned}$$

where K_ν denotes the modified Bessel function of the third kind with index ν. The domain of variation of the parameters is $0 \leq |\beta| < \alpha$, $\mu, \lambda \in \mathbb{R}$ and $\delta > 0$.

Thus GH distributions are characterized by the five parameters $(\lambda,\alpha,\beta,\delta,\mu)$. Alternative parameters used in the literature are

$$\begin{aligned} &\zeta = \delta\sqrt{\alpha^2-\beta^2}, \qquad \rho = \beta/\alpha, \\ &\xi = (1+\zeta)^{-1/2}, \qquad \chi = \xi\rho, \\ &\overline{\alpha} = \alpha\delta, \qquad \overline{\beta} = \beta\delta. \end{aligned}$$

These alternative parameters are scale- and location-invariant, i.e. they do not change under affine transformations $Y = aX + b$ with $a \neq 0$ of a given variable X. Let η (resp. ν) denote the expectation (resp. the variance) of the distribution given by the density above. It can be shown that the mapping $(\lambda,\alpha,\beta,\delta,\mu) \to (\lambda,\xi,\chi,\nu,\eta)$ is bijective. Therefore $(\lambda,\xi,\chi,\nu,\eta)$ where $0 \leq |\chi| < \xi < 1$ represents a parametrization with a rather intuitive interpretation. λ is a class parameter, ξ and χ are invariant shape parameters whereas ν (resp. η) are the variance (resp. the expectation), i.e. they are the scale and the location parameter.

The properties of the Bessel function K_λ (Abramowitz and Stegun (1968)) allow one to find simpler expressions for the Lebesgue density if $\lambda \in {}^1\!/_2\,\mathbb{Z}$. For $\lambda = 1$ we get the hyperbolic distribution which is characterized by the fact that the log-density is a hyperbola. This subclass has the simplest representation of all GH laws, which is favourable from a numerical point of view. For $\lambda = -1/2$ we get the normal inverse Gaussian (NIG) distribution. This subclass is closed under convolution for fixed parameters α and β. See Eberlein and Keller (1995), Eberlein, Keller, and Prause (1998), Barndorff-Nielsen (1998), Barndorff-Nielsen and Prause (1999) for statistical results concerning the subclasses of hyperbolic (resp. NIG distributions).

2 Estimation of Densities

We estimate generalized hyperbolic, hyperbolic and normal inverse Gaussian distributions from daily as well as from high-frequency data. The algorithm and the results concerning German stock prices and NYSE indices are described in detail in Prause (1997, 1999). Analogous results are obtained for the DAX, the German stock index (see Figure 1). Let $(S_t)_{t\geq 0}$ be the price process for a given financial instrument. We define the *return* of this instrument for a given time interval Δt, e.g. one trading day, as

$$X_t = \log S_t - \log S_{t-\Delta t}. \qquad (1)$$

Thus the return during n periods is the sum of the one period returns. The numerical estimates for the GH distribution and the subclasses are given in Table 1.

Table 1. *Generalized hyperbolic parameter estimates for the daily returns of the DAX from December 15, 1993 to November 26, 1997. The parameter λ is fixed for the estimation of the hyperbolic and the NIG distribution.*

	λ	α	β	δ	μ	Log-Likelihood
GH	−2.018	46.82	−24.91	0.0163	0.00336	3138.28
Hyperbolic	1	158.87	−29.02	0.0059	0.00374	3135.15
NIG	−0.5	105.96	−26.15	0.0112	0.00348	3137.33

Figure 1 (top) provides a typical plot of empirical and estimated GH densities. The plot of the densities shows that the GH, hyperbolic and NIG distributions are more peaked and have more mass in the tails than the normal distribution. Consequently they are much closer to the empirical distribution of asset returns. Although the difference between GH, hyperbolic and the NIG distribution is small, it is clear that the generalized hyperbolic distributions are superior to those of the subclasses.

Value-at-Risk (VaR) has become a major tool in the modelling of risk inherent in financial markets. Essentially VaR is defined as the potential loss given a level of probability $\alpha \in (0,1)$

$$P[X_t < -\mathrm{VaR}_\alpha] = \alpha. \qquad (2)$$

The quantity defined here has to be transformed in the proper way if one wants to express VaR in currency units. The plot of VaR as a function of α could also be used to visualize the tail behaviour of distributions. Note, that the concept of VaR applied only for a single α is not satisfactory: VaR does not identify extreme risks appearing with a probability smaller than α. Figure 1 (bottom) shows that the tails of the generalized hyperbolic distributions are heavier than the tails of the normal distribution and therefore VaR computed

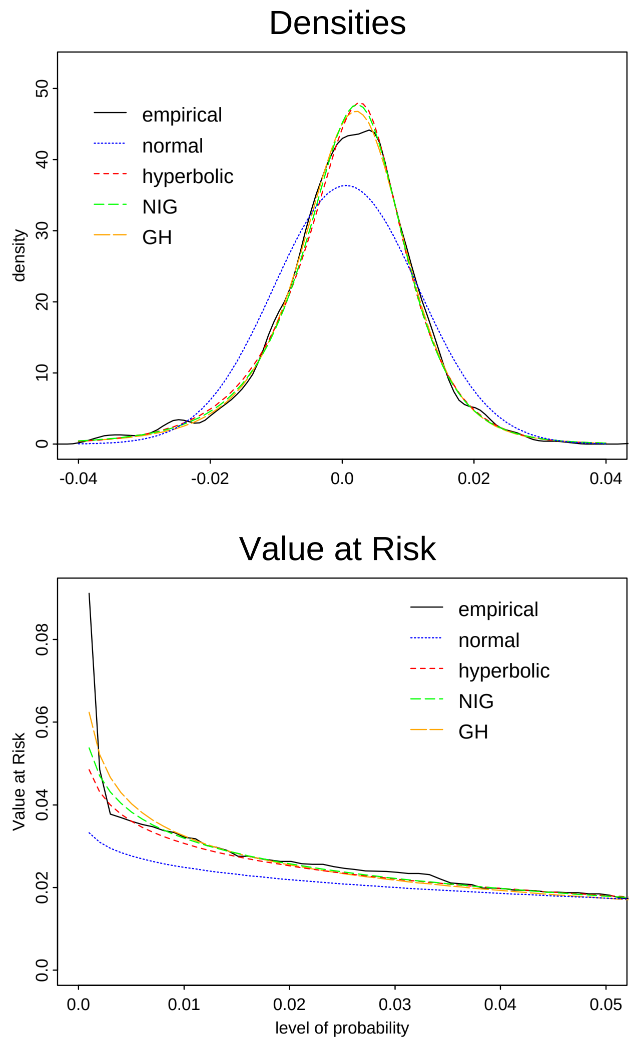

Fig. 1. *DAX from December 15, 1993 to November 26, 1997, daily prices at 12:00h, IBIS data (Karlsruher Kapitalmarktdatenbank).*

parametrically for the GH distribution and its subclasses is closer to the empirically observed Value-at-Risk.

In the global foreign exchange (FX) market it is particularly important to look at price movements on an intraday basis. Many traders close their positions over night and try to make a profit from intraday trading only. Therefore we examine 6 hours returns of USD/DEM exchange rates from the HFDF96 dataset provided by Olsen & Associates. See also J.P. Morgan

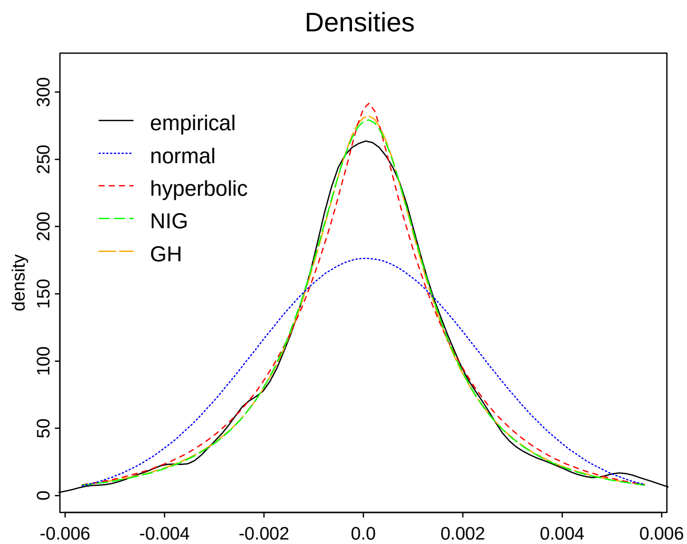

Fig. 2. *USD/DEM exchange rate from January 1 to December 31, 1996, 6 hours returns, HFDF96 dataset (Olsen & Associates, Zürich).*

and Reuters (1996, p. 65) for some remarks concerning the leptokurtosis of daily USD/DEM returns. For high-frequency data we follow Guillaume, Dacorogna, Davé, Müller, Olsen, and Pictet (1997) in the definition of the log-price

$$p(t_i) = [\log p_{ask}(t_i) + \log p_{bid}(t_i)]/2 \tag{3}$$

and the corresponding return

$$r(t_i) = p(t_i) - p(t_i - \Delta t). \tag{4}$$

We estimate the GH parameters for the increments $r(t_i)$ after removing all zero-returns. Although this is only a provisional approach to focus on time periods where trading takes place, the results as plotted in Figure 2 provide a clear picture: The excellent fit of generalized hyperbolic distributions and the typical difference to the normal distribution observed for daily returns is repeated for high-frequency data (see also Barndorff-Nielsen and Prause (1999)).

3 The Generalized Hyperbolic Model

We follow Eberlein and Keller (1995) in the design of the price process $(S_t)_{t\geq 0}$ and the derivation of an option pricing formula. First we construct the driving process. Generalized hyperbolic distributions are infinitely divisible (Barndoff-Nielsen and Halgreen (1977)). Therefore they generate a Lévy process $(X_t)_{t\geq 0}$, i.e. a process with stationary and independent increments, such that the distribution of X_1 and thus of $X_t - X_{t-1}$ is generalized hyperbolic. We call this process $(X_t)_{t\geq 0}$ the *generalized hyperbolic Lévy motion.* It depends on the five parameters $(\lambda, \alpha, \beta, \delta, \mu)$ and is purely discontinuous. This property follows from the explicit form of the Lévy-Khintchine representation of the characteristic function of generalized hyperbolic distributions which is given in the appendix. The exponent consists only of a drift term and the integral representing the jumps, but has no Gaussian term $-{}^{c}\!/_{2}\, u^2$. The new model for the price process itself is defined by

$$S_t = S_0 \exp(X_t). \tag{5}$$

Let us emphasize that (5) is only the basic model which replaces the classical geometric Brownian motion introduced by Osborne and Samuelson. During the last 40 years this classical Gaussian model, which can also be defined via the diffusion equation

$$dS_t = S_t(\mu dt + \sigma dB_t), \tag{6}$$

has been generalized and refined in many directions. In its most sophisticated generalization (see e.g. Bakshi, Cao, and Chen (1997)) jumps are added through a Poisson process, the constant volatility σ is replaced by a diffusion process driven by a different Brownian motion and a stochastic interest rate is considered, which is typically given in the form of a Cox-Ingersoll-Ross model. Taking correlations between the various driving processes into acount one has to consider more than ten parameters. Calibration of such a model is not an easy task.

Essentially every extension which has been considered for the geometric Brownian motion can be applied to the exponential Lévy model (5) as well. The extension we consider to be crucial and which improves the model considerably is stochastic volatility. In (5) this can be done by writing X_t in the form $\mu t + \sigma L_t$ where $(L_t)_{t\geq 0}$ is a standardized Lévy process, that is one with mean zero and variance one. In this form σ can be replaced by any of the standard models for stochastic volatility such as diffusion models or the Ornstein-Uhlenbeck-based models considered by Barndorff-Nielsen and Shephard (2001) and Nicolato and Prause (1999) or any member of the ARCH and GARCH-family. A detailed discussion of this issue supported by a number of empirical results will be given in Eberlein, Kallsen, and Kristen (2001).

The key property of our model–besides its simplicity–is that taking log-returns in (5) one obtains the corresponding increment of the driving Lévy process $(X_t)_{t\geq 0}$. For time intervals of length 1 its distribution is by construction the generating generalized hyperbolic distribution. Thus the model produces for time intervals of length 1 exactly that distribution which one

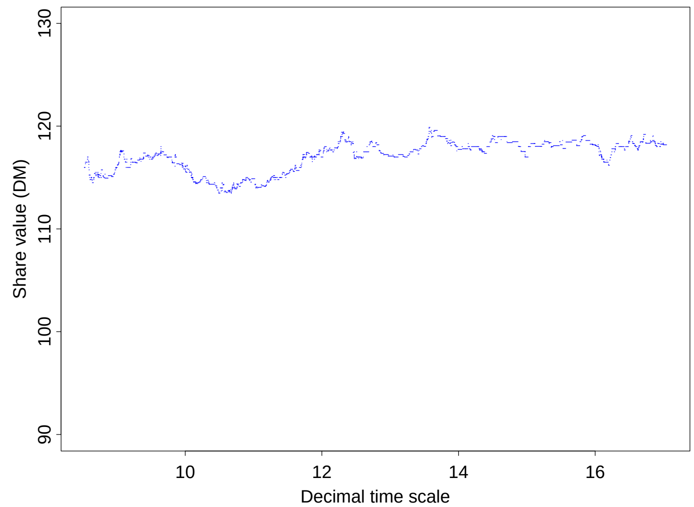

Fig. 3. *Siemens, Xetra data from August 28, 1998.*

gets from fitting data. If, for example, the underlying data set consists of daily prices, one trading day in real time corresponds to a time interval of length 1 in the model. It is not only this reproduction of observed distributions which makes (5) attractive, the model is also consistent in a much deeper sense. If one calibrates a model using daily data a natural question is whether the distribution produced by the model for a weekly horizon is close to the distribution one obtains by fitting the corresponding weekly data. This turns out to be the case to a certain degree of accuracy. Of course the same should hold if one goes in the other direction, namely from daily to intraday hourly data. Recall that the classical Gaussian model produces normal log-returns along any time interval Δt. Detailed results on this consistency property in both directions will appear in a forthcoming joint paper with Fehmi Özkan.

In this context let us clarify that the generalized hyperbolic model (5) does not have anything in common with the hyperbolic diffusion model introduced by Bibby and Sørensen (1997) and discussed further in Rydberg (1999). The latter is a classical diffusion model with completely different statistical as well as path properties.

The price process (5) has purely discontinuous paths as has the driving Lévy process. In order to give the reader an idea of what the paths of such a process look like, we show in Figure 3 a sample of the intraday price behaviour of stocks. To model the microstructure of asset prices, purely discontinuous processes are more appropriate than the classical or the hyperbolic diffusion processes with continuous paths.

Since we are in an incomplete setting, we have to select a specific equivalent martingale measure. Arbitrage free prices are obtained as expectations under these measures (Delbaen and Schachermayer (1994)). Note, that it is possible to obtain every price in the full no-arbitrage interval by chosing the proper equivalent martingale measure (Eberlein and Jacod (1997)). We choose the Esscher equivalent martingale measure P^θ given by

$$\mathrm{d}P^\theta = \exp\bigl(\theta X_t - t \log M(\theta)\bigr)\mathrm{d}P. \tag{7}$$

The parameter θ is the solution of $r = \log M(\theta+1) - \log M(\theta)$ where M is the moment generating function given in the Appendix and r is the constant interest rate. The equation for θ ensures that the discounted price process is in fact a P^θ-martingale. Chan (1999) remarked that in a model very similar to the exponential Lévy model (5), the Esscher transform is the minimal martingale measure in the sense of Föllmer and Schweizer (1991). A much deeper motivation for the choice of this particular martingale measure came out of several recent papers, where via duality theory it was shown that the choice of a minimal martingale measure corresponds to maximizing expected utility. More precisely, taking the Esscher transform corresponds to maximizing utility with respect to the power utility function $u(x) = x^p/p$. One among several good references for this application of duality theory to finance is Goll and Rüschendorf (2000).

Following the arbitrage pricing theory, the price of an option with time to expiration T and payoff function $H(S_T)$ is given by $e^{-rT}\mathrm{E}^\theta[H(S_T)]$. In particular, for a call option with strike K whose payoff is $H(S_T) = (S_T - K)^+$ we obtain the price formula

$$S_0 \int_\gamma^\infty \mathrm{gh}^{*T}(x;\theta+1)\,\mathrm{d}x - e^{-rT}K \int_\gamma^\infty \mathrm{gh}^{*T}(x;\theta)\,\mathrm{d}x, \tag{8}$$

where $\gamma = \ln(K/S_0)$ and $\mathrm{gh}^{*t}(\,\cdot\,;\theta)$ is the density of the distribution of X_t under the risk-neutral measure. The density $\mathrm{gh}^{*t}(\,\cdot\,)$ of the t-fold convolution of the generalized hyperbolic distribution can be computed by applying the Fourier inversion formula to the characteristic function. In the case of NIG distributions one should of course use the property that this subclass is closed under convolution.

Figure 4 shows that the difference of the generalized hyperbolic option prices to those from the Black-Scholes model resembles the W-shape which was observed for hyperbolic option prices. Note that for options with short maturities the W-shape is more pronounced in the case of the NIG and the GH model.

4 Rescaling of Generalized Hyperbolic Distributions

For the computation of implicit volatilities in the GH model we need to rescale the generalized hyperbolic distribution while keeping the shape fixed.

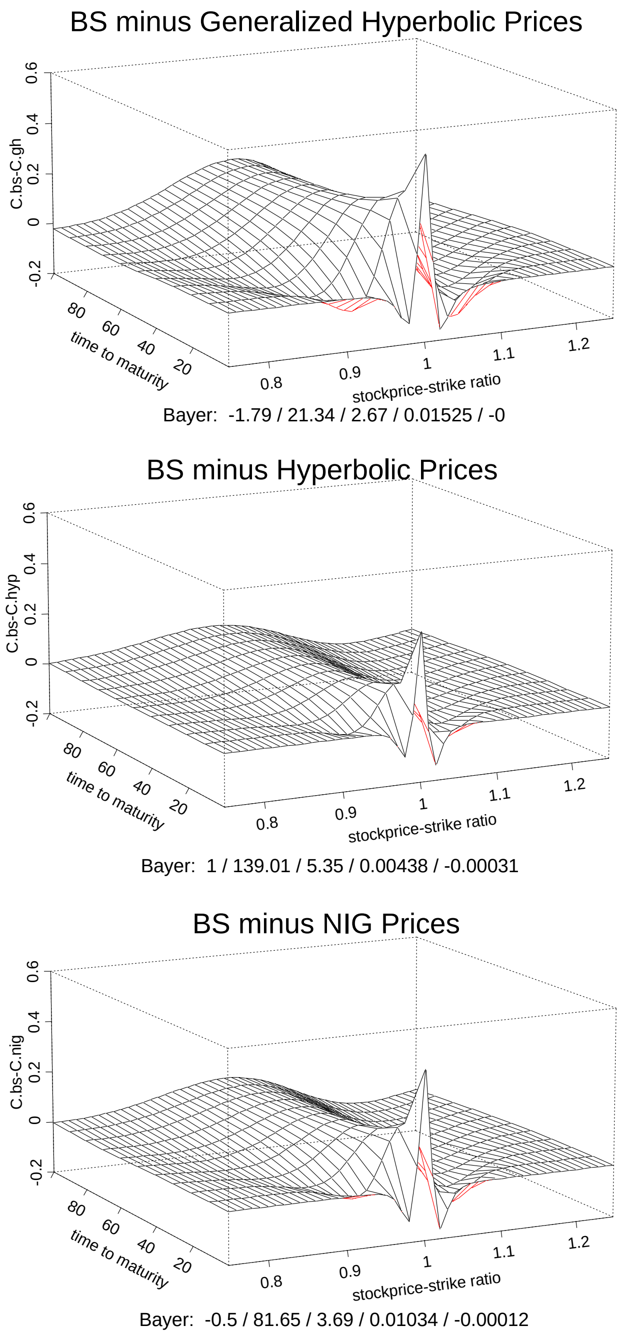

Fig. 4. *Black-Scholes minus GH prices (Bayer parameters, strike K=1000).*

An analogous problem occurs when computing GH option prices for a given volatility, e.g. a volatility estimated from historical stock returns. In this section we also give some insights into the structure of GH distributions. The rescaling of generalized hyperbolic distributions is based on the following property concerning scale- and location-invariance.

Lemma 1. *The terms λ, $\alpha\delta$ and $\beta\delta$ are scale- and location-invariant parameters of the univariate generalized hyperbolic distribution. The very same holds for the alternative parametrizations (ζ, ρ) and (ξ, χ).*

Proof. According to Blæsild (1981) a linear transformation $Y = aX + b$ of a GH distributed variable X is again GH-distributed with parameters $\lambda^+ = \lambda$, $\alpha^+ = \alpha/|a|$, $\beta^+ = \beta/|a|$, $\delta^+ = \delta|a|$ and $\mu^+ = a\mu + b$. Obviously $\alpha^+\delta^+ = \alpha\delta$ and $\beta^+\delta^+ = \beta\delta$.

A consequence of Lemma 1 is that the variance of the generalized hyperbolic distribution has the linear structure $\mathrm{Var}[X_1] = \delta^2 C_\zeta$ in δ^2 where C_ζ depends only on the shape, i.e. the scale- and location-invariant parameters (Barndorff-Nielsen and Blæsild (1981)). Therefore one can also use δ as a scaling parameter. To rescale the distribution for a given variance $\widehat{\sigma}^2$ one obtains the new $\widetilde{\delta}$ as

$$\widetilde{\delta} = \widehat{\sigma}\left[\frac{K_{\lambda+1}(\widehat{\zeta})}{\widehat{\zeta}K_\lambda(\widehat{\zeta})} + \frac{\widehat{\beta}^2}{\widehat{\alpha}^2 - \widehat{\beta}^2}\left(\frac{K_{\lambda+2}(\widehat{\zeta})}{K_\lambda(\widehat{\zeta})} - \left(\frac{K_{\lambda+1}(\widehat{\zeta})}{K_\lambda(\widehat{\zeta})}\right)^2\right)\right]^{-1/2} \tag{9}$$

where $(\widehat{\alpha}, \widehat{\beta}, \widehat{\delta})$ and consequently $\widehat{\zeta}$ are estimated from a longer time series. To fix the shape of the distribution while rescaling with a new $\widetilde{\delta}$, one has to change the other parameters in the following way

$$\widetilde{\lambda} = \widehat{\lambda}, \quad \widetilde{\alpha} = \frac{\widehat{\alpha}\,\widehat{\delta}}{\widetilde{\delta}}, \quad \widetilde{\beta} = \frac{\widehat{\beta}\,\widehat{\delta}}{\widetilde{\delta}} \text{ and } \widetilde{\mu} = \widehat{\mu}. \tag{10}$$

Note, that the term in the square brackets is scale- and location-invariant. In order to value German stock options we use shape parameters estimated from stock prices from January 1, 1988 to May 24, 1994 and we rescale the estimated generalized hyperbolic distributions while feeding in volatility estimates from shorter time periods.

Figure 5 shows the densities and the corresponding log-densities of hyperbolic distributions. In the first row we fix the shape estimated from Bayer stock prices and rescale the distribution as described in (10). The second row of Figure 5 reveals that ζ describes the kurtosis of the distribution. For increasing ζ the density becomes less peaked and converges to the Gaussian distribution. Log-densities give some insight into the tail behaviour of the density. The log-density of the hyperbolic distribution is a hyperbola whereas the normal log-density is a parabola. Therefore hyperbolic distributions possess substantially heavier tails than the normal distribution. Nevertheless, in contrast to those of stable distributions, excluding the normal distribution, all moments of GH distributions do exist.

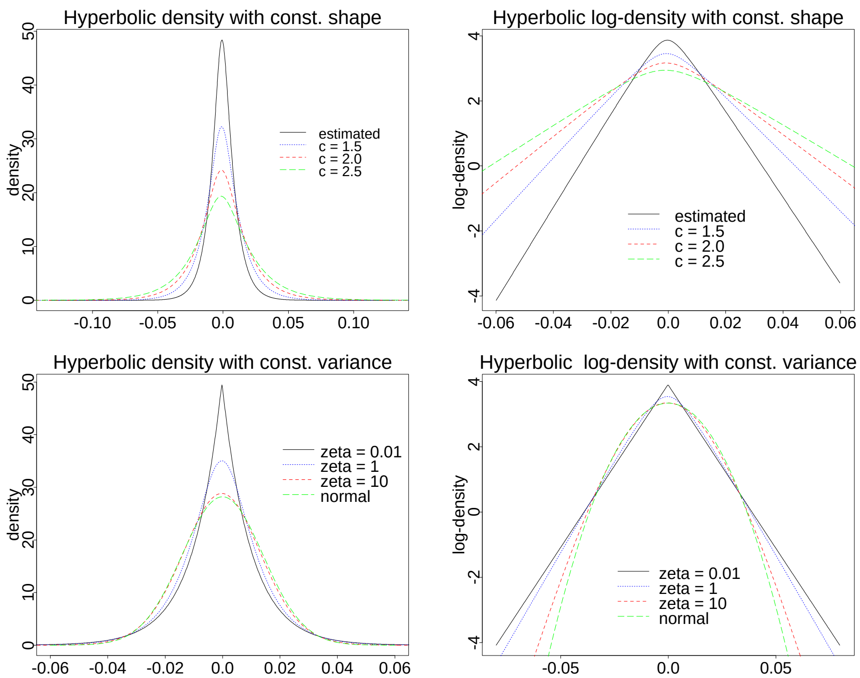

Fig. 5. *Rescaled hyperbolic densities ($\delta' = c\,\widehat{\delta}$ with constant shape parameters $\widehat{\zeta} = 0.608$, $\widehat{\rho} = 0.0385$ estimated from Bayer stock prices) and hyperbolic densities with constant variance and different shapes.*

5 Smile Reduction

The comparison of generalized hyperbolic prices with Black-Scholes prices in Section 3 hints at the possibility to correct the well-known smiles which appear in Black-Scholes implicit volatilities. Implicit volatilities are computed from observed option prices by inverting the corresponding pricing formula with respect to the volatility parameter. Usually all parameters necessary for option pricing are known to traders except the volatility. In the GH model we rely on the rescaling mechanism described in Section 4 to obtain the volatility parameter. In this section we compute the implicit volatilities. The study is based on intraday option and stock market data of Bayer, Daimler Benz, Deutsche Bank, Siemens and Thyssen from July 1992 to August 1994. The option data set contains all trades reported by the Deutsche Terminbörse (since 1998 Eurex Germany) during the period above. The preparation of the data sets is described in detail in Eberlein, Keller, and Prause (1998, Chapter IV). The latter article includes also a discussion of implicit volatilities in the hyperbolic model and of the different approaches to reduce the smile.

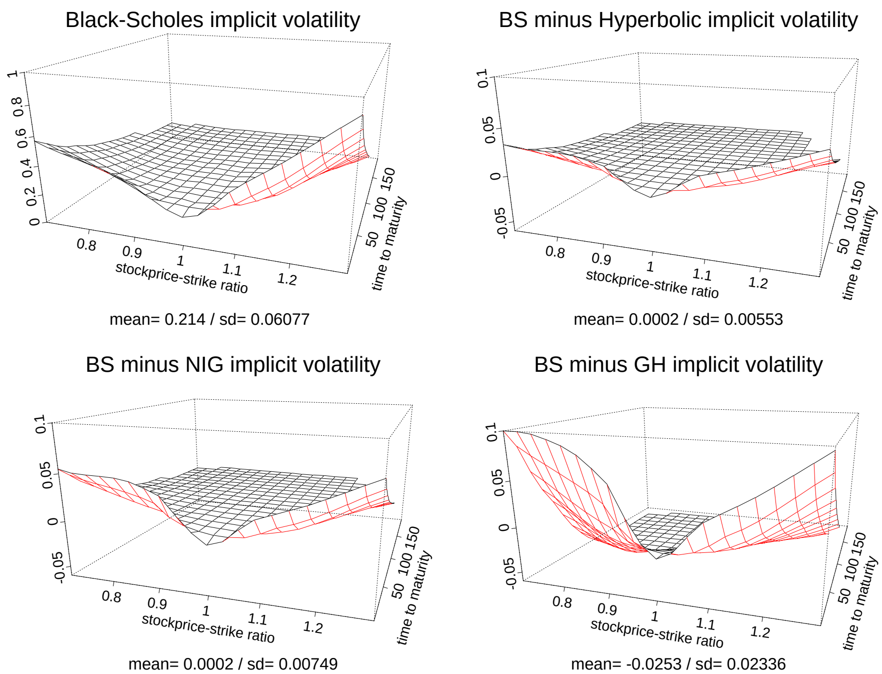

Fig. 6. *Black-Scholes implicit volatilities and comparison of the implicit volatilities of Black-Scholes, hyperbolic, NIG, and GH prices (Daimler Benz calls from July 1992 to August 1994, 62504 observations).*

Implicit volatilities in the Black-Scholes model typically follow a pattern denoted as *smile*, i.e. they are low for options at the money and the highest implicit volatilities are observed for options with short maturities in and out of the money. Figure 6 (top left) shows the implicit volatilities of Daimler Benz calls in the Black-Scholes model. To compare these with implicit volatilities in the GH model, we computed the differences and plotted them in Figure 6. The pattern reflects the W-shapes from Figure 4. Obviously we observe a more pronounced correction of the smile effect in the GH model–due to the heavier tails of the distribution.

A different approach to analyse the smile behaviour of a particular option pricing model is to fit a linear model for the implicit volatilities of the form

$$\sigma_{\mathrm{Imp},i} = b_0 + b_1 T_i + b_2 (\rho_i - 1)^2 / T_i + e_i, \tag{11}$$

where e_i is the random error term, ρ_i the stockprice-strike ratio S/K and i the number of the trade in the option data set. The cross-term $(\rho - 1)^2/T$ reflects the degression of the smile effect with increasing time to maturity T. Table 2 shows the regression coefficients for the Black-Scholes, the GH models and the respective symmetric centered versions of each. The values of

Table 2. *Fitted coefficients for call options from July 1992 to August 1994. SC marks the results for the symmetric centered versions of the models.*

		b_0	b_1	b_2	R^2
Daimler Benz	Black-Scholes	0.2177	−0.00029	40.53	0.5416
	Hyperbolic	0.2186	−0.0003	36.89	0.4972
	Hyperbolic SC	0.2184	−0.000293	36.33	0.4951
	NIG	0.2191	−0.000305	35.11	0.4746
	NIG SC	0.2189	−0.000296	34.48	0.4716
	GH	0.2207	−0.000321	32.81	0.4378
	GH SC	0.2201	−0.000306	31.98	0.4343

the coefficient for $(\rho - 1)^2/T$ are smaller for hyperbolic, NIG and GH prices compared to those from Black-Scholes prices. Hence these new models reduce the smile effect. The largest correction is observed for the symmetric centered GH model.

6 Multivariate Generalized Hyperbolic Distributions

In the previous sections we have discussed univariate generalized hyperbolic distributions as the basic ingredient for a stock price model focussing on option pricing. We shall now look into the estimation of multivariate GH distributions and its application to risk measurement.

Definition 2. For $x \in \mathbb{R}^d$, the d-dimensional *generalized hyperbolic distribution* (GH_d) is defined by its Lebesgue density, which is given by

$$\mathrm{gh}_d(x) = a_d \, \frac{K_{\lambda-d/2}\big(\alpha\sqrt{\delta^2 + (x-\mu)'\Delta^{-1}(x-\mu)}\,\big)}{\big(\alpha^{-1}\sqrt{\delta^2 + (x-\mu)'\Delta^{-1}(x-\mu)}\,\big)^{d/2-\lambda}} \, \exp(\beta'(x-\mu)),$$

$$a_d = a_d(\lambda,\alpha,\beta,\delta,\Delta) = \frac{\big(\sqrt{\alpha^2 - \beta'\Delta\beta}\,/\delta\big)^{\lambda}}{(2\pi)^{d/2}K_\lambda\big(\delta\sqrt{\alpha^2 - \beta'\Delta\beta}\,\big)}$$

The parameters have the following domain of variation[1]: $\lambda \in \mathbb{R}$, $\beta, \mu \in \mathbb{R}^d, \delta > 0$, $\beta'\Delta\beta < \alpha^2$. The positive definite matrix $\Delta \in \mathbb{R}^{d\times d}$ has a determinant $|\Delta| = 1$.

For $\lambda = (d+1)/2$ we obtain the *multivariate hyperbolic* and for $\lambda = -1/2$ the *multivariate normal inverse Gaussian* distribution. Generalized hyperbolic distributions are symmetric iff $\beta = (0, \dots, 0)'$.

[1] We omitted the limiting distributions obtained at the boundary of the parameter space; see e.g. Blæsild and Jensen (1981).

Blæsild and Jensen (1981) introduced alternative parameters ζ, π, S where $\zeta = \delta\sqrt{\alpha^2 - \beta'\Delta\beta}$, $\pi = \beta\Delta^{1/2}(\alpha^2 - \beta'\Delta\beta)^{-1/2}$ and $S = \delta^2\Delta$. Generalized hyperbolic distributions are closed under forming marginals, conditioning and affine transformations (Blæsild (1981)). For the mean and the variance of $X \sim \mathrm{GH}_d$ one obtains

$$\mathrm{E}X = \mu + \delta R_\lambda(\zeta)\pi\Delta^{1/2}, \tag{12}$$

$$\mathrm{Var}\, X = \delta^2 \left(\zeta^{-1}R_\lambda(\zeta)\Delta + S_\lambda(\zeta)\big(\pi\Delta^{1/2}\big)'\big(\pi\Delta^{1/2}\big)\right), \tag{13}$$

where in order to simplify notation we introduced $R_\lambda(x) = K_{\lambda+1}(x)/K_\lambda(x)$ and $S_\lambda(x) = \big[K_{\lambda+2}(x)K_\lambda(x) - K_{\lambda+1}^2(x)\big]/K_\lambda^2(x)$.

A maximum likelihood estimation of all parameters in higher dimensions is computationally too demanding since the number of parameters $3 + d(d+5)/2$ increases rapidly with the number of dimensions. Therefore we propose a simplified algorithm for symmetric GH distributions which allows for an efficient estimation also in higher dimensions. The first step of the estimation follows a method of moments approach: we estimate the sample mean $\widehat{\mu} \in \mathbb{R}^d$ and the sample dispersion matrix Σ using canonical estimators. Since $\pi = 0$ in the symmetric case, $\mathrm{E}X = \mu$, and from (13) we get the following estimate for Δ

$$\widehat{\Delta} = \frac{\zeta}{\delta^2 R_\lambda(\zeta)}\,\Sigma. \tag{14}$$

Consequently we compute $\widehat{\Delta}$ by norming the sample dispersion matrix such that $|\Delta| = 1$. The second step is to compute

$$y_i = (x_i - \widehat{\mu})'\widehat{\Delta}^{-1}(x_i - \widehat{\mu}) \tag{15}$$

from observations $x_i \in \mathbb{R}^d$, $1 \leq i \leq n$. Then the log-likelihood function is given as

$$\begin{aligned} L(x; \lambda, \alpha, \delta) = n\Big(\lambda \log(\alpha/\delta) - \frac{d}{2}\log(2\pi) - \log K_\lambda(\delta\alpha)\Big) \\ + \sum_{i=1}^n \log K_{\lambda-(d/2)}\big(\alpha\sqrt{\delta^2 + y_i}\,\big) + \Big(\lambda - \frac{d}{2}\Big)\sum_{i=1}^n \log\big(\sqrt{\delta^2 + y_i}/\alpha\big). \end{aligned} \tag{16}$$

The last step is to maximize this log-likelihood function with respect to $(\lambda, \alpha, \delta)$. We have developed efficient estimation algorithms for hyperbolic and NIG distributions, i.e. for fixed $\lambda = 1$ and $\lambda = -1/2$. In the case of arbitrary λ one may encounter numerical problems due to extremely small values of the Bessel functions K_λ. As in the univariate case the log-likelihood function simplifies for $\lambda \in {}^1\!/_2\,\mathbb{Z}$. For NIG distributions, i.e. $\lambda = -1/2$, the number of Bessel functions K_λ which have to be computed for the log-likelihood function is reduced by one. In the case of hyperbolic and hyperboloid distributions we have to compute only one Bessel function instead of $n + 1$. Since the evaluation of Bessel functions is the time-consuming part of the third step, computation is much simpler for hyperbolic distributions. For fixed λ it is also possible to estimate only ζ in the second step. Nevertheless, we have chosen to estimate the covariance structure in the first "method of moments" step and the parameters (α, δ) characterizing the kurtosis and the scale in the likelihood step.

For a price process $S_t \in \mathbb{R}^d$ we define *relative returns* $x_t \in \mathbb{R}^d$ by

$$x_t^{(i)} = \left[S_t^{(i)} - S_{t-\Delta t}^{(i)}\right] / S_{t-\Delta t}^{(i)} \approx \log S_t^{(i)} - \log S_{t-\Delta t}^{(i)}, \quad 1 \le i \le d, \tag{17}$$

which are approximated by the log-returns defined in (1). The motivation to choose this definition is that the return of a portfolio described by a vector $h \in \mathbb{R}^d$ is then simply given by $h'x_t$. See J.P. Morgan and Reuters (1996, Section 4.1) for a discussion of temporal and cross-section aggregation of asset returns.

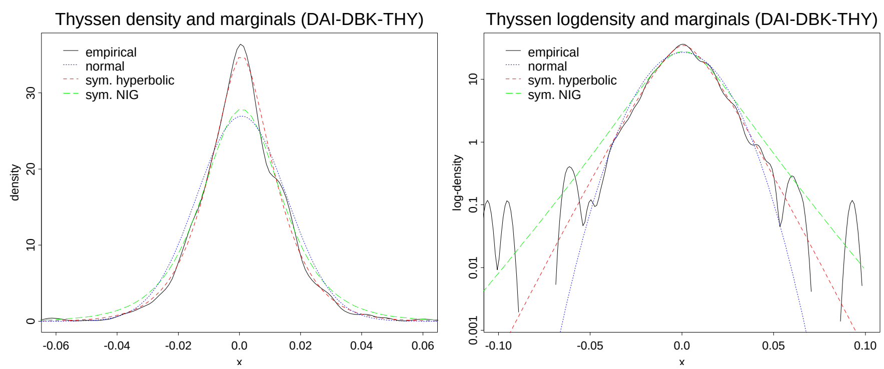

Fig. 7. *Marginal density for Thyssen obtained from the 3-dimensional estimate for Daimler Benz–Deutsche Bank–Thyssen.*

The marginal densities of the GH distributions can be derived using a theorem of Blæsild (1981). Typically we obtain the pattern shown in Figure 7 for the densities and log-densities: The marginal distributions of hyperbolic and NIG distributions are closer to the empirical distribution than the normal distribution. In the center, marginals of hyperbolic distributions are closer to the empirical distribution but in the tails, marginals of NIG distributions provide a better fit.

7 Market Risk Measurement

Let us start with a general result on densities.

Theorem 1. *Let X be a d-dimensional random variable with symmetric generalized hyperbolic distribution, i.e. with $\beta = (0,\ldots,0)'$, and let $h \in \mathbb{R}^d$ where $h \neq (0,\ldots,0)'$. The distribution of $h'X$ is univariate generalized hyperbolic* $\mathrm{GH}_d(\lambda^\times, \alpha^\times, \beta^\times, \delta^\times, \mu^\times)$*, where $\lambda^\times = \lambda$, $\alpha^\times = \alpha|h'\Delta h|^{-1/2}$, $\beta^\times = 0$, $\delta^\times = \delta|h'\Delta h|^{1/2}$ and $\mu^\times = h'\mu$.*

Proof. Let $h_1 \neq 0$ without loss of generality. Apply Theorem Ic) of Blæsild (1981) with

$$A = \begin{pmatrix} h_1 & h_2 & \cdots & h_d \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{18}$$

Then project the d-dimensional GH distribution onto the first coordinate using Theorem Ia).

The latter theorem may be used to calculate risk measures for a portfolio of d assets with investments given by a vector $h \in \mathbb{R}^d$. As an example we look at a portfolio consisting of three German stocks: Daimler Benz, Deutsche Bank and Thyssen from January 1, 1988 to May 24, 1994. We choose $h = (1,1,1)'$ and show the empirical density of the returns $h'x_t$ of the portfolio in Figure 8. The previous theorem gives the corresponding densities obtained from the d-dimensional estimates of symmetric hyperbolic and symmetric NIG distributions. Figure 8 shows also the direct estimate of the univariate GH distribution from $h'x_t$.

The densities and log-densities in Figure 8 indicate that symmetric GH distributions enable one to perform more precise modelling of the return distribution of the portfolio. As a consequence one can get more realistic risk measures than the traditional ones based on the normal distribution. Figure 9 shows a risk measure over a 1-day horizon with respect to a level of probability $\alpha \in (0,1)$, namely the shortfall which we define as

$$\mathrm{Shortfall}_{\alpha,t} = -\mathrm{E}\big[h'x_t | h'x_t < q(\alpha)\big], \tag{19}$$

where $q : [0,1] \to \mathbb{R}$ is the corresponding quantile function.

Note that the shortfall goes clearly beyond the concept of VaR because it takes into account the extreme negative returns. The log-density of the empirical distribution in Figure 8 shows the magnitude of the negative returns of multi-asset portfolios in relation to the more frequent small returns.

The Basle Committee on Banking Supervision (1995, IV.23) has proposed a *backtesting* procedure to test the quality of Value-at-Risk estimators. We follow this procedure to compare standard VaR estimation approaches with VaR estimators based on GH distributions.[2] After computing the VaR for each day in the time period from January 1, 1989 to May 24, 1994 we count

[2] The Basle Committee on Banking Supervision (1995, IV.3) recommends a holding period of 10 days. Nevertheless, we consider a 1-day horizon only because

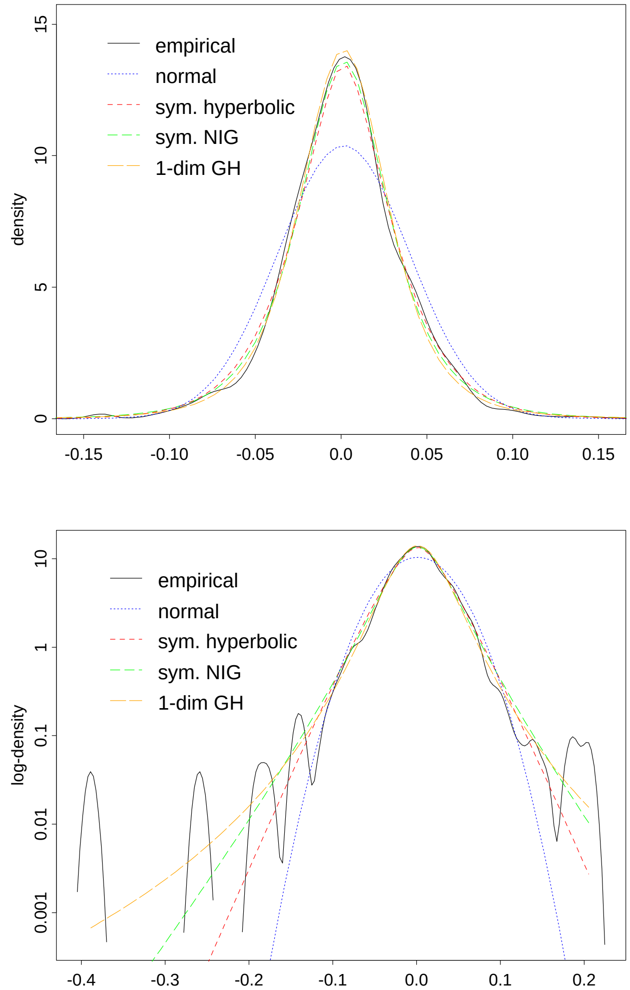

Fig. 8. *Distribution of the returns of a portfolio consisting of Daimler Benz, Deutsche Bank, and Thyssen (equal weights).*

the observed losses greater than the Value-at-Risk. Since Value-at-Risk is essentially a quantile, the percentage of excess losses should correspond to the level of probability α. One standard method to compute VaR is to simulate

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the increased number of returns in the observation period allows more accurate statistical results. Note, that we would not upscale a 1-day VaR by multiplying it with $\sqrt{10}$. Instead we would use the distribution corresponding to 10 days in our model.

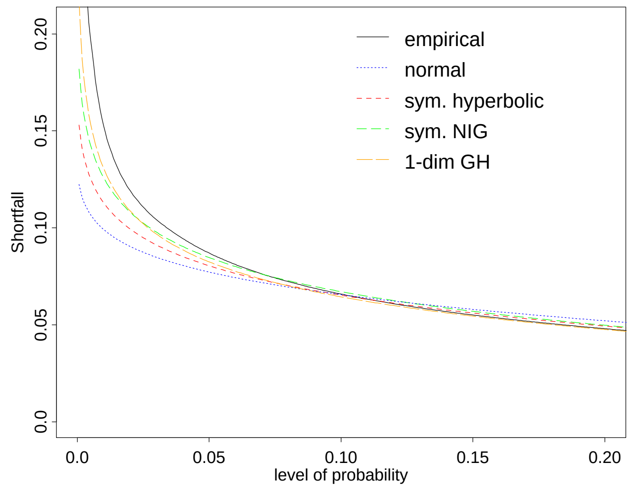

Fig. 9. *Shortfall for Daimler Benz–Deutsche Bank–Thyssen (portfolio with equal weights).*

the return of the portfolio by the preceding 250 observed returns and to take the quantile of this empirical distribution. This is historical simulation. A second simulation technique proposed to forecast VaR is Monte Carlo simulation. It is computationally intensive for large portfolios. We have not applied this method here because distinct differences to the Variance-Covariance approach are only obtained for nonlinear portfolios (Bühler, Korn, and Schmidt (1998)). However, a full valuation approach based on GH distributions, for instance for portfolios with derivative contracts, is easily implemented (Prause (1999)). The mixture representation of GH distributions allows to generate random numbers efficiently. We also apply the Variance-Covariance approach which is based on the multivariate normal distribution.

The results given in Table 3 show that the standard estimators for Value-at-Risk underestimate the risk of extreme losses on the relevant level of 1%. This effect is visible in the percentage of excess losses in the historical simulation. In the Variance-Covariance approach we observe too high values for the level of probability $\alpha = 1\%$ and too small values for $\alpha = 5\%$. The percentages of realized losses greater than VaR are closer to the level of probability in both cases $\alpha = 1\%$ and $\alpha = 5\%$ for the symmetric hyperbolic and the symmetric NIG distribution.

An approach similar to the rescaling mechanism proposed above for the univariate case is to estimate the shape from a longer time period and to use an up-to-date covariance matrix Σ. This allows one to incorporate the risk of extreme events, even if they do not occur in the preceding 250 trading days, which is the minimum time period proposed by the Basle Committee (1995). Therefore we have to choose a subclass, i.e. a parameter $\lambda \in \mathbb{R}$, and to fix a long-term estimate for ζ. We compute the matrix S in the alternative parametrization by

$$S = \delta^2 \Delta = \frac{\zeta}{R_\lambda(\zeta)} \Sigma. \tag{20}$$

A further refinement is possible by choosing an appropriate estimate for the covariance matrix. We select the multivariate IGARCH model of Nelson (1990) in which variance $\sigma_{1,t}^2$ and covariance $\sigma_{12,t}^2$ are given by

$$\sigma_{1,t}^2 = (1-\lambda) \sum_{t \geq 1} \lambda^{t-1} (r_t - \bar{r}), \tag{21}$$

$$\sigma_{12,t}^2 = (1-\lambda) \sum_{t \geq 1} \lambda^{t-1} (r_{1,t} - \bar{r}_1)(r_{2,t} - \bar{r}_2), \tag{22}$$

where $0 < \lambda < 1$ is a decay factor, $r_t, r_{1,t}, r_{2,t}$ returns of financial assets and $\bar{r}, \bar{r}_1, \bar{r}_2$ the corresponding mean values. To allow for a comparison, we have used the decay factor $\lambda = 0.94$ applied in J.P. Morgan and Reuters (1996) for daily returns.

Table 3. *Ex post evaluation of risk measures: percentage of losses greater than VaR. Each trading day the Value-at-Risk for a holding period of one day is estimated from the preceding 250 trading days (Daimler Benz, Deutsche Bank, and Thyssen from January 1, 1989 to May 24, 1994, Investment of 1DM in each asset).*

VaR Estimation Method	$\alpha = 1\%$	$\alpha = 5\%$
Historical Simulation	2.08	5.79
Variance-Covariance	1.63	4.45
RiskMetrics / IGARCH	1.34	4.75
Symmetric hyperbolic	1.48	4.9
Symmetric NIG	1.26	4.75
Symmetric hyperbolic, long-term ζ	1.26	4.45
Symmetric NIG, long-term ζ	1.04	4.9
Hyperbolic IGARCH, long-term ζ	1.11	4.82
NIG IGARCH, long-term ζ	1.11	5.34
1-dimensional Hyperbolic	1.41	4.9
1-dimensional NIG	1.41	4.97

Finally we propose to reduce the risk measurement problem to one dimension by computing quantiles for the return $h'x_t$ of the whole portfolio.

We estimate hyperbolic and NIG distributions and derive the corresponding quantiles.

The results of the study for a linear portfolio are shown in Table 3. Taken together, the use of a long-term shape parameter incorporates the possibility of extreme events, even if there was no crash in the preceding 250 trading days, whereas the GH-IGARCH approach describes the volatility clustering observed in financial markets. This yields more accurate results for GH-based models in the ex-post evaluation of the risk measures.

8 Conclusion

In the first part of this paper we presented generalized hyperbolic distributions resp. their subclasses and estimation results concerning daily as well as high-frequency returns. The greater flexibility of this class of distributions allows an almost perfect fit to empirical asset return distributions. Based on the Lévy processes generated by these infinitely divisible distributions we introduced in section 3 the generalized hyperbolic model as a new way to describe asset prices. It is a rather natural model, since it reproduces exactly those distributions which one observes in the data. An option pricing formula can be derived using the Esscher transform as in Eberlein and Keller (1995). Using the rescaling mechanism of generalized hyperbolic distributions we analyzed implicit volatilities and prices obtained in the GH model. We observed a correction of the smile effect in the GH model.

Risk measures are used in financial institutions with two objectives. Internally they give the management a possibility to allocate risk capital.

> Setting limits in terms of risk helps business managers to allocate risk to those areas which they feel offer the most potential, or in which their firms' expertise is greatest. This motivates managers of multiple risk activities to favor risk reducing diversification strategies.[3]

On the other hand regulators as well as the management want to reduce the probability of default. Therefore they set limits to the exposure to market risk relative to the capital of the firm.

Is Value-at-Risk the adequate measure for this purpose? Quantile-based methods like VaR have the disadvantage that they do not consider losses occuring with a probability below a given level of probability. Stress testing offers a partial solution to this problem focussing on extreme scenarios. To quantify risk properly one has to forecast the whole profit and loss distribution. Regulators should use other risk measures than VaR as well. In this context we also would like to mention the axiomatic concept of coherent risk measures developed by Artzner, Delbaen, Eber, and Heath (1999).

In the last two sections we have shown that it is possible to estimate generalized hyperbolic distributions in an efficient way and to construct more

[3] J.P. Morgan and Reuters (1996, p. 33), see also Chart 3.1.

accurate risk measures for multivariate price processes. Symmetric hyperbolic and symmetric NIG distributions are characterized by the covariance matrix and a shape parameter. This simple structure allows a further sophistication of GH risk measures by fixing a long-term shape parameter, which describes the probability of rare events, and choosing a short-term estimate for the covariance matrix. A study in accordance with the backtesting concept required by the Basle Committee on Banking Supervision reconfirms the excellent results concerning VaR estimation for multivariate price processes. Moreover, we have shown that generalized hyperbolic distributions are also the proper building block for risk measures beyond VaR.

Acknowledgement
We thank Deutsche Börse AG, Frankfurt for a number of data sets concerning stock and option prices. We also used IBIS data from the Karlsruher Kapitalmarktdatenbank and the high-frequency data set HFDF96 provided by Olsen & Associates, Zürich.

Appendix

Moment Generating and Characteristic Function

Lemma A.1. *The moment generating function of the generalized hyperbolic distribution is*

$$M(u) = e^{u\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda/2}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+u)^2}\,)}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2}\,)}, \quad |\beta+u|<\alpha.$$

Lemma A.2. *The characteristic function of the generalized hyperbolic distribution is*

$$\phi(u) = e^{i\mu u}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\lambda/2}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+iu)^2}\,)}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2}\,)}.$$

Theorem A.1. *The Lévy-Khintchine representation of $\phi(u)$ is*

$$\ln\phi(u) = iu\mu + \int\big(e^{iux}-1-iux\big)g(x)\mathrm{d}x, \tag{23}$$

with density

$$g(x) = \frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{\exp(-\sqrt{2y+\alpha^2}\,|x|)}{\pi^2 y\big(J_\lambda^2(\delta\sqrt{2y})+Y_\lambda^2(\delta\sqrt{2y}\,)\big)}\,\mathrm{d}y + \mathbf{1}_{\{\lambda\ge 0\}}\,\lambda e^{-\alpha|x|}\right). \tag{24}$$

Here J_λ and Y_λ denote Bessel functions of the first and second kind.

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