PART II

INTEREST RATE MODELS

1

STATIC INTEREST RATE MODELS

1.1 - FITTING THE TS OF INTEREST RATES

- Fundamental Asset Pricing Formula
- Spot Rates
- Recovering the Term Structure
- Direct Methods: **Bootstraping** and **Interpolation**
- Indirect Methods: **Deterministic interest rate models**
- Spline Methods

FUNDAMENTAL ASSET PRICING FORMULA

$$P_0 = \sum_{t=1}^{T} \frac{CF_t}{(1+R_{0,t})^t} = \sum_{t=1}^{T} CF_t p(0,t)$$

- *R(0,t)* is the discount rate (the yield in the case of a bond)
- *p(0,t)* is the discount factor (equal to 1 if cash-flows are received at t=0)

Two main questions:

- **Q1:** Where do we get the *p(0,t)* or *R(0,t)* from?
- **Q2:** Do we use the equation to obtain bond prices or implied discount factors/discount rates?

Answers:

- **Q1:** Where do we get the *p(0,t)* or *R(0,t)* from?

  Any relevant information concerning how to price a financial asset must be primarily obtained from **market sources**

  – Discount factor *p(0,t)* is the price of a t-Bond with unitary face value and maturity *t*

  – Spot rate $R_{0,t}$ is the annualized rate of a pure discount bond:

$$\frac{1}{(1+R_{0,t})^t} = p\left(0,t\right)$$

- **Example**: Getting Spot rates from Bond Prices:

  – Consider a two-year pure discount bond that trades at 92€.

  – The two-year spot rate $R_{0,2}$ is:

$$92 = \frac{100}{(1+R_{0,2})^2}; \quad R_{0,2} = 4.26\%$$

- In the real world, zero-coupon bonds are not abundant.

- However, we are still able to compute spot rates from market information.

- **Q2:** Do we use the equation to obtain bond prices or implied discount factors/discount rates?

It depends on the situation!

- Roughly speaking, one would like to use the price of primitive securities as given, and derive implied discount factors or discount rates from them.
- Then, one may use that information (more specifically the term structure of discount rates) to price any other security.
- This is known as relative pricing.

1.1.1 - Direct Methods

- Consider two securities (nominal 100€):
  - 1-year pure discount bond selling at 95€.
  - 2-year 8% bond selling at 99€, with annual coupon payments.
- 1-year spot rate:

$$95 = \frac{100}{(1+R_{0,1})}; \;\; R_{0,1} = 5.26\%$$

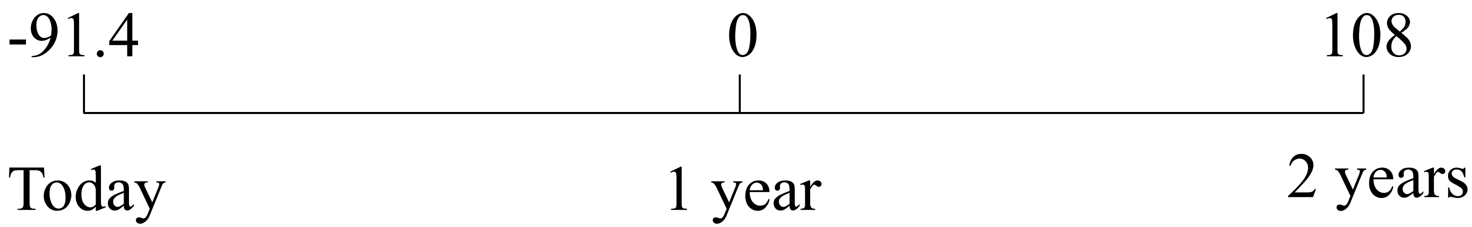
- 2-year spot rate:

$$99 = \frac{8}{(1+R_{0,1})} + \frac{108}{(1+R_{0,2})^2} = \frac{8}{1.0526} + \frac{108}{(1+R_{0,2})^2};$$

$$R_{0,2} = 8.7\%$$

- We may also "**construct**" a two year pure discount bond with the same yield:

- Two components:
  - Buy the 2-year bond
  - Sell 0.08 units of the 1-year bond
- Cost: $99 - (0.08) \times 95 = 91.4$



- Two-year rate is:

$$91.4 = \frac{108}{(1 + R_{0,2})^2}; \; R_{0,2} = 8.7\%$$

EXAMPLE

	Coupon	Maturity (year)	Price
Bond 1	5	1	101
Bond 2	5.5	2	101.5
Bond 3	5	3	99
Bond 4	6	4	100

- Solve the following system:

  101 = 105 p(0,1)

  101.5 = 5.5 p(0,1) + 105.5 p(0,2)

  99 = 5 p(0,1) + 5 p(0,2) + 105 p(0,3)

  100 = 6 p(0,1) + 6 p(0,2) + 6 p(0,3) + 106 p(0,4)

- and obtain

  p(0,1)=0.9619, p(0,2)=0.9114, p(0,3)=0.85363, p(0,4)= 0.7890

  R(0,1)=3.96%, R(0,2)=4.717%, R(0,3)=5.417%, R(0,4)=6.103%

FINDINGS

- If you can find different bonds with coincident cash-flow dates and one of them only has one remaining cash-flow date, then you can directly get the spot rates – **Bootstrapping method**
- Notice that these rates are not the yield (except for the shortest bond) and consequently they do not face their consistency problems.
- Therefore, we have a single spot rate for each maturity and the yield curve may have any shape.
- A usual practical way to estimate the yield curve involves the employment of interbank money market rates for several maturities, as illustrated in the following example.

FINDINGS

Maturity	ZC
Overnight	4.40%
1 month	4.50%
2 months	4.60%
3 months	4.70%
6 months	4.90%
9 months	5.00%
1 year	5.10%

	Coupon	Maturity (years)	Price
Bond 1	5%	1 y and 2 m	103.7
Bond 2	6%	1 y and 9 m	102
Bond 3	5.50%	2 y	99.5

- 1 year and 2 months rate
  x=5.41%
- 1 year and 9 months rate
  y= 5.69%
- 2 year rate
  z= 5.69%

$$103.7 = \frac{5}{(1+4.6\%)^{1/6}} + \frac{105}{(1+x)^{1+1/6}}$$

$$102 = \frac{6}{(1+5\%)^{9/12}} + \frac{6}{(1+y)^{1+9/12}}$$

$$99.5 = \frac{5.5}{(1+5.1\%)^{1}} + \frac{105.5}{(1+z)^{2}}$$

INTERPOLATION - LINEAR

- If we don't have all market information required to get spot rates for the same maturities, we may opt for linear interpolation.
- Assuming that we know discount rates for maturities t_1 and t_2, the rate for maturity t, being $t_1 < t < t_2$, corresponds to the weighted average of the adjacent rates, being the weights higher for the maturity closer to t.

$$R(0,t) = \frac{(t_2 - t)R(0,t_1) + (t - t_1)R(0,t_2)}{(t_2 - t_1)}$$

- Linear interpolations provide good proxies for near maturities.
- For distant maturities, the shape of the resulting yield curve tends to be kinked.
- By definition, linear interpolation doesn't allow to get estimates for maturities longer than those observed.

INTERPOLATION – POLYNOMIAL

- **Polynomial interpolations of the interest rates** allow to obtain smoother yield curves.
- For instance, if we consider a cubic polynomial interpolation, one can estimate the full term structure just by knowing the spot rates for 4 maturities.
- Therefore, if $R(0, t_1)$, $R(0, t_2)$, $R(0, t_3)$ and $R(0, t_4)$ are known, one can solve the following system in order to the four coefficients of the 3rd order polynomial:

$$\begin{cases} R(0,t_1) = at_1^3 + bt_1^2 + ct_1 + d \\ R(0,t_2) = at_2^3 + bt_2^2 + ct_2 + d \\ R(0,t_3) = at_3^3 + bt_3^2 + ct_3 + d \\ R(0,t_4) = at_4^3 + bt_4^2 + ct_4 + d \end{cases}$$

- If one uses more than 4 spot rates, these coefficients are estimated by econometric techniques, e.g. ordinary least squares (as the functions are linear in the coefficients).

EXAMPLE

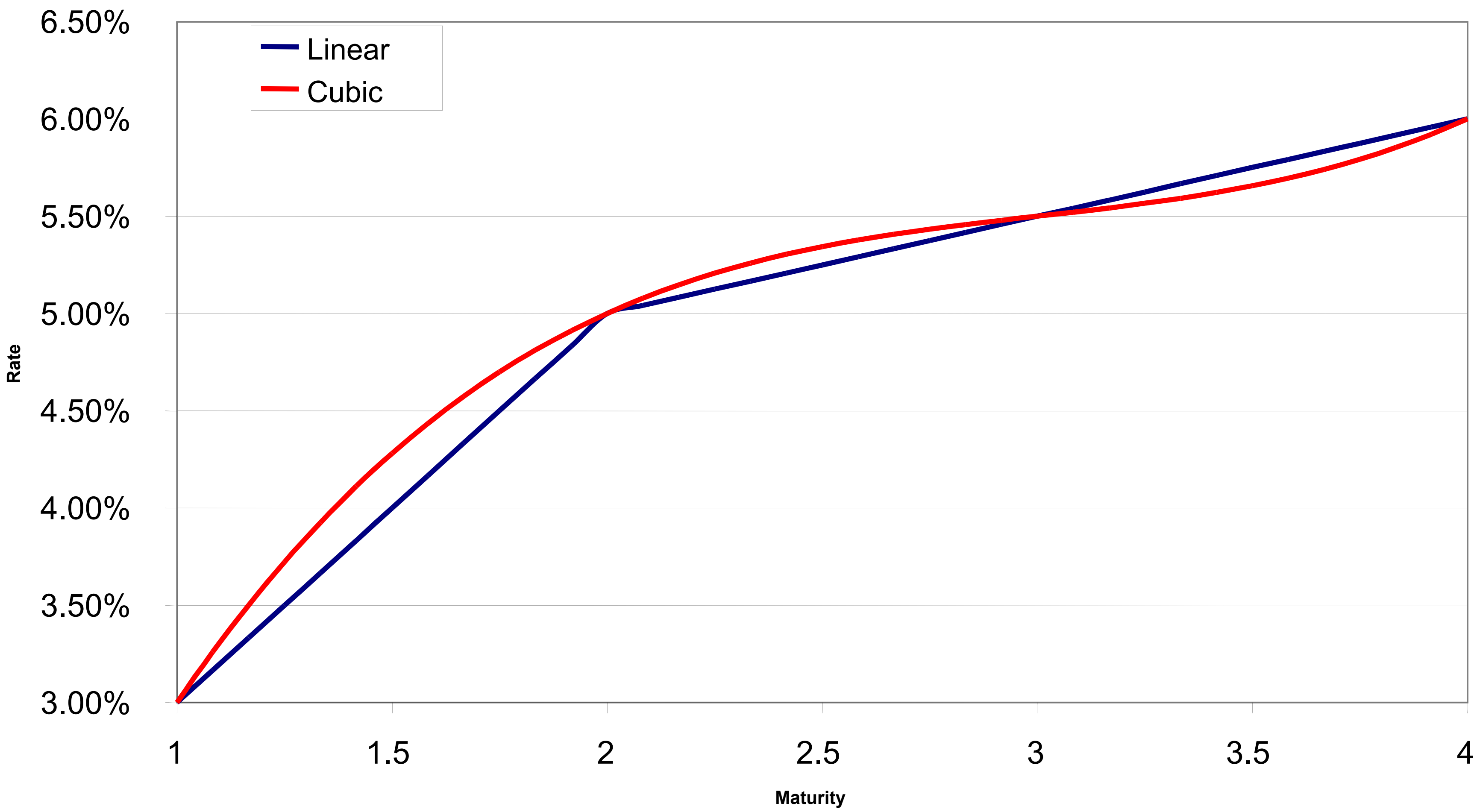
- The calculation of *a*, *b*, *c* and *d* allows to obtain the spot rate for any maturity *t*: $R(0,t) = at^3 + bt^2 + ct + d$
- Assuming the following rates are known:
  - R(0,1) = 3%
  - R(0,2) = 5%
  - R(0,3) = 5.5%
  - R(0,4) = 6%
- Goal - Compute the 2.5 year rate:

$R(0,2.5) = a \times 2.5^3 + b \times 2.5^2 + c \times 2.5^1 + d = 5.34375\%$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3\% \\ 5\% \\ 5.5\% \\ 6\% \end{pmatrix} = \begin{pmatrix} 0.0025 \\ -0.0225 \\ 0.07 \\ -0.02 \end{pmatrix}$$

As R = T x A, being R, T and A the column vector of interest rates, the matrix with the several factors depending on *t* and *A* the column vector with the coefficients *a* to *d* => A $=T^{-1} \times R$

ILLUSTRATION: POLYNOMIAL VERSUS LINEAR



FINDINGS

- The resulting spot curve using 3rd order polynomial methods tends to be too irregular, namely when:
  - one uses it to estimate a rate for a maturity much higher than the maximum maturity used to calculate the polynomial coefficients
  - the difference between two consecutive maturities is too large.

- One way to mitigate this problem is to use **polynomial splines**, a technique that allows to have slight differences in the polynomial specification in different maturity buckets.

1.1.2 - Spline Methods

Polynomial Functions

- Discount factors (*p*) as polynomial functions of the maturity (*s*):

$$p(s) = \begin{cases} p_0(s) = d_0 + c_0 s + b_0 s^2 + a_0 s^3, s \in [0,5] \\ p_5(s) = d_1 + c_1 s + b_1 s^2 + a_1 s^3, s \in [5,10] \\ p_{10}(s) = d_2 + c_2 s + b_2 s^2 + a_2 s^3, s \in [10,20] \end{cases}$$

- Imposing continuity constraints and given the fact that the discount factor for zero maturity is 1, the number of parameters are reduced:

$$p_0^{(i)}(5) = p_5^{(i)}(5)$$

$$p_{10}^{(i)}(10) = p_5^{(i)}(10) \qquad i = 0,1,2$$

$$p_0(0) = 1$$

POLYNOMIAL SPLINES

- One of the most used methodology of polynomial splines was developed by McCulloch (1971; 1975).

- The maturity spectrum is divided in *k-2* intervals, with *k-3* vertices, considering that the discount function can be defined by a cubic function, adding a factor (spline) to the 3^{rd} order component in *k-3* intervals (the polynomial in the 1^{st} interval will not have this additional factor, being the discount function given just by the 3^{rd} order polynomial).

- No. of parameters = k

POLYNOMIAL SPLINES

- The discount function can be written as follows :

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + \sum_{h=1}^{k-3} a_{4,h+1}(t - t_h)^3 \cdot D_h(t)$$

where $D_h(t)$ for h=1,2,..., k-3 are functions defined on the basis of the vertices of the intervals, as follows:

$D_h(t)$ =0, if $t<t_h$, $D_h(t)$=1, if $t \geq t_h$, for h=1,....,k-3.

- The discount function is continuous ⇔ for all vertices, the values for the discount :

$$d(t) = a_0 + \sum_{h=1}^{k} a_h g_h(t)$$

POLYNOMIAL SPLINES

- **How to choose the number of parameters/intervals and the vertices:**

  – McCulloch proposes *k* = square root of the number of observations (bonds), rounded to the nearest integer, with the vertices chosen to ensure all intervals have the same No. observations (or the difference between the No. observations in each interval is not higher than 1).

  – Alternative methodology - fixing the vertices of the intervals in maturity dates corresponding to the maturities in which the market is traditionally “divided”: 1, 3, 5 and 10 years.

POLYNOMIAL SPLINES

- If the vertices of the intervals correspond to the maturities in which the market is traditionally "divided" - 1, 3, 5 and 10 years – we have:
  - No. Intervals: k-2 = 5
  - No. Vertices: k-3 = 4

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + \sum_{h=1}^{k-3} a_{4,h+1}(t - t_h)^3 \cdot D_h(t)$$

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + a_{4,2}(t-1)^3 \cdot D_1(t) + a_{4,3}(t-3)^3 \cdot D_2(t)$$
$$+ a_{4,4}(t-5)^3 \cdot D_3(t) + a_{4,5}(t-10)^3 \cdot D_4(t)$$

$D_1(t) = 0$, if *t<1*, $D_1(t) = 1$, if $t \geq 1$ $D_2(t) = 0$, if *t<3*, $D_2(t) = 1$,if $t \geq 3$
$D_3(t) = 0$, if *t<5*, $D_3(t) = 1$, if $t \geq 5$ $D_4(t) = 0$, if *t<10*, $D_4(t) = 1$, if $t \geq 10$

POLYNOMIAL SPLINES

- The method of polynomial splines provides us better estimates in sample, i.e. up to the longest observed maturity, comparing to polynomial functions.
- However, the estimation problems outside the sample remain, as the discount function tends to assume irregular shapes from the longest maturity onwards, and it may even become negative.
- Whenever the yield curve assumes complex shapes, the use of a high number of parameters leads the estimated curve to adjust excessively to outliers => yield curve becomes even more irregular.
- This is particularly inconvenient if the objective is, as it usually happens, the estimation of the term structure of interest rates for a fixed or *standardised* range of maturities, or to calculate forward rates.
- Therefore, more complex specifications will be required.

1.1.3 - Deterministic Methods

- 3 steps:
  - **Step 1**: select a set of *K* bonds with prices P^j paying cash-flows $F^j(t_i)$ at dates $t_i > t$
  - **Step 2**: select a deterministic interest rate model for the functional form of the discount factors $p(t,t_i;\beta)$, or the discount rates $R(t,t_i;\beta)$, where β is a vector of unknown parameters, and generate prices.

$$\hat{P}^j(t) = \sum_{i=1}^{N} CF^j(t_i) p(t,t_i;\beta) = \sum_{i=1}^{N} CF^j(t_i) e^{-\left(t_j - t\right) R(t,t_i;\beta)}$$

  - **Step 3**: estimate the parameters β as the ones making the theoretical prices as close as possible to market prices:

$$\beta = \arg\min \sum_{j=1}^{K} \left(\hat{P}^j(t) - P^j(t) \right)^2$$

- Despite the drawback that they **lack theoretical underpinnings**, the BIS reported that 9 out of 13 central banks which report their curve estimation methods to the BIS use deterministic Interest Rate Models (BIS (2005), “Zero-coupon yield curves: technical documentation”, BIS Papers, No 25, Monetary and Economic Department, October 2005).

- According to this study, most central banks reporting data have adopted either the Nelson and Siegel (1987) model or the extended version suggested by Svensson (1994). Exceptions are Canada, Japan, Sweden, UK and the US, which all apply variants of the “smoothing splines” method.

- Deterministic interest rate models are also widely used among market practitioners.

- Key advantages:
  - Parsimonious models, i.e. do not involve many parameters
  - Ensure stable functions
  - Adjust to many possible shapes of the TS
  - Some parameters have economic interpretation

RECAP AND NOTATION

Spot rate curve as a function of maturity (s_m) = simple average of instantaneous forward rates:

$$s_m = \frac{1}{m} \cdot \int_0^m {}_{\mu} f_0 d\mu$$

Discount curve:

$$d_m = e^{-s_m \cdot m}$$

Forward rate for time to settlement *m* and for maturity *n* (with discrete compounding):

$${}_m f_n = \left[\frac{(1+s_{m+n})^{m+n}}{(1+s_m)^m} \right]^{\frac{1}{n}} - 1$$

Given that the spot rate can be obtained from the discount function $s_t = \left(\frac{1}{d_t} \right)^{1/t} - 1$

We obtain

$${}_m f_n = \left[\frac{d_m}{d_{m+n}} \right]^{\frac{1}{n}} - 1$$

1.1.3.1 - Nelson Siegel

- Nelson and Siegel (1987) proposed to fit the term structure using a flexible, smooth parametric function.
- They demonstrated that their proposed model is capable of capturing many of the typically observed shapes that the yield curve assumes over time.
- The resulting Nelson-Siegel forward curve can be assumed to be the solution to a second order differential equation with equal roots for spot rates, corresponding to a 3 unobserved factor model (as pointed out in Diebold and Li (2005)):

$$ {}_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau)} + \beta_2 \cdot \left[(m/\tau) \cdot e^{(-m/\tau)} \right] $$

$$ s_m = \beta_0 + (\beta_1 + \beta_2) \cdot \left[1 - e^{(-m/\tau)} \right] / (m/\tau) - \beta_2 \cdot \left[e^{(-m/\tau)} \right] $$

$$ d_m = e^{\left[-\beta_o m - (\beta_1 + \beta_2)\tau \left(1 - e^{-\frac{m}{\tau}} \right) + \beta_2 m \cdot e^{\left(-\frac{m}{\tau} \right)} \right]} $$

β_0 : level parameter - the long-term rate

$\beta_0 + \beta_1$: short-term rate

β_1 : (-) slope parameter

β_2: curvature parameter

τ : influences the speed of convergence of the curve towards the asymptotic value.

$\left(1 - \frac{\beta_1}{\beta_2}\right)\tau$: point of inflection of the slope of the forward curve

$\left(2 - \frac{\beta_1}{\beta_2}\right)\tau$: point of inflection of the concavity of the forward curve

1.1.3.2 – DIEBOLD, PIAZZESI AND RUDEBUSCH

- Some authors argue that the NS model has to many parameters to be estimated.
- Litterman and Scheinkman (1991)* show that the variation in interest rates can be explained by a small number of underlying common factors, typically up to **three**, interpreted as level, slope and curvature.
- The 1st factor explains 89,5% of the total variance of returns, the 2nd factor for 8,5% and the 3rd for the remaining 2%.
- For this reason, Diebold, Piazzesi, and Rudebusch (2005)*2 examine a **2-factor Nelson-Siegel model**, even though they recognize that more than 2 factors will "be needed in order to obtain a close fit to the entire yield curve at any point in time", e.g. for pricing derivatives.

* Litterman, Robert and José Scheinkman (1991), "Common Factors Affecting Bond Returns", Journal of Fixed Income.

*2 Diebold, Francis X., Monika Piazzesi and Glenn D. Rudebusch (2005), "Modeling Bond Yields in Finance and Macroeconomics", American Economic Review, 95, pp. 415-420.

DPR (2005) TWO-FACTOR MODEL

- Compared to the three-factor Nelson-Siegel model, the two-factor model only contains the level and slope factor => only 3 parameters have to be estimated:

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left[\frac{1-\exp\left(-\frac{\tau}{\lambda_t}\right)}{\left(\frac{\tau}{\lambda_t}\right)}\right]$$

- Diebold, Piazzesi, and Rudebusch (2005) argue that since the first two principal components explain nearly all variation in interest rates, a two-factor model may suffice to forecast the term structure.

1.1.3.3 – Björk and Christensen

- Although NS model can capture a wide range of shapes, it cannot handle all the shapes that the term structure assumes over time.

- As an attempt to remedy this problem, several more flexible NS specifications have been proposed in the literature to improve the fit to more complex shapes, namely with multiple inflection points, introducing additional factors and parameters.

- One of those models was developed by Bjork, T. and Christensen B.J. (1999): ”Interest rate dynamics and consistent forward rate curves”. Mathematical Finance.

1.1.3.3 – Björk and Christensen

- Björk and Christensen (1999) propose to add a fourth factor to the approximating forward curve

$$f_t(\tau) = \beta_{1,t} + \beta_{2,t} \exp\left(-\frac{\tau}{\lambda_t}\right) + \beta_{3,t} \left(\frac{\tau}{\lambda_t}\right) \exp\left(-\frac{\tau}{\lambda_t}\right) + \beta_{4,t} \exp\left(-\frac{2\tau}{\lambda_t}\right)$$

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t} \left[\frac{1-\exp\left(-\frac{\tau}{\lambda_t}\right)}{\left(\frac{\tau}{\lambda_t}\right)}\right] + \beta_{3,t} \left[\frac{1-\exp\left(-\frac{\tau}{\lambda_t}\right)}{\left(\frac{\tau}{\lambda_t}\right)} - \exp\left(-\frac{\tau}{\lambda_t}\right)\right]$$

$$+ \beta_{4,t} \left[\frac{1-\exp\left(-\frac{2\tau}{\lambda_t}\right)}{\left(\frac{2\tau}{\lambda_t}\right)}\right]$$

The fourth component, resembles the second component as it also mainly affects short-term maturities.
The difference is that it decays to zero at a faster rate.

BC (1999) FOUR-FACTOR - PROPERTIES

- The factor $\beta_{4,t}$ can be interpreted as a second slope factor.
- As a result, the Björk Christensen model captures the slope of the term structure by the (weighted) sum of $\beta_{2,t}$ and $\beta_{4,t}$.
- The instantaneous short rate in this case is given by

$$y_t(0) = \beta_{1,t} + \beta_{2,t} + \beta_{4,t}$$

1.1.3.4 – BLISS

- A second option to make the Nelson-Siegel more flexible is through relaxing the restriction that the slope and curvature component should be governed by the same decay parameter τ.

- Bliss (1997) estimates the term structure of interest rates with the three-factor model that allows for two different decay parameters τ_1 and τ_2.

- The forward curve is then given by

$$_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau) \cdot e^{(-m/\tau_2)} \right]$$

- The Bliss functional form for the yield, then becomes

$$s_m = \beta_0 + \beta_1 \cdot \left[1 - e^{(-m/\tau_1)}\right] / \left(m / \tau_1\right) + \beta_2 \cdot \left[\left[1 - e^{(-m/\tau_2)}\right] / \left(m / \tau_2\right) - \left[e^{(-m/\tau_2)}\right]\right]$$

b0 : level parameter - the long-term rate

b0 + b1: short-term rate

b1 : (-) slope parameter

b2: curvature parameter

τ_1 and τ_2 : influences the speed of convergence of the curve towards the asymptotic value.

1.1.3.5 – SVENSSON

- A popular term-structure estimation method among central banks (see BIS, 2005) is the four-factor Svensson (1994) model.
- Svensson (1994) proposes to increase the flexibility and fit of the Nelson-Siegel model by adding a second hump-shape factor with a separate decay parameter.
- This feature is specially relevant when fitting the short segment of the yield curves, following disturbances in the money markets that lead to curves with two local optima (two points of inflection of the slope) or with two points of inflection of the concavity.
- As the NS method admits the existence of only one point of inflection in the slope and concavity, the fit in the short segment of the yield curve turns out to be very poor under such circumstances.

- The resulting four-factor forward curve is given by:

$$_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau_1) \cdot e^{(-m/\tau_1)}\right] + \beta_3 (m/\tau_2) e^{(-m/\tau_2)}$$

- Thus, the spot rate will be given by the following expression:

$$s_m = b_0 + b_1 \cdot \left[1 - e^{(-m/\tau_1)}\right] / (m/\tau_1) +$$
$$+ b_2 \cdot \left\{\left[1 - e^{(-m/\tau_1)}\right] / (m/\tau_1) - e^{(-m/\tau_1)}\right\} +$$
$$+ b_3 \cdot \left\{\left[1 - e^{(-m/\tau_2)}\right] / (m/\tau_2) - e^{(-m/\tau_2)}\right\}$$

β_0 : level parameter - the long-term rate

$\beta_0 + \beta_1$: short-term rate

β_1 : (-) slope parameter

β_2, β_3: curvature parameters

τ_1, τ_2 : influences the speed of convergence of the curve towards the asymptotic value.

SVENSSON MODEL - PROPERTIES

- The Svensson method is more adequate to estimate the term structure of interest rates for monetary policy purposes, given its higher adjustment capacity in the segment of the shorter maturities.
- However, when the yield curve assumes in the short segment simple shapes the estimation by the NS method seems preferable since it is more parsimonious.
- In fact, the NS model is a restricted version of the Svensson model with the restriction $\beta_3 = 0$ and/or $\tau_2 \to 0$. Thus, using a likelihood ratio test we can test the null hypothesis corresponding to those restrictions:

$$H_0: \ \beta_0 = \beta_1 = \ldots = \beta_q = 0 \qquad \lambda = -2 \cdot (\ln v - \ln v^*) \approx \chi^2(q)$$

where: v = likelihood function of the adjustment with restrictions; $v*$ = likelihood function of the adjustment without restrictions; q = number of restrictions.

SVENSSON MODEL - PROPERTIES

- In this case, *v* corresponds to the likelihood function of the NS model, while *v** is the likelihood function of the Svensson model.

- Thus, if the logarithm of the likelihood function of the Svensson model is large enough (i.e., is statistically above that of the NS model), the Svensson model will be selected.

- A potential problem with the Svensson model is that it is highly non-linear which can make the estimation of the model difficult (see Bolder and Stréliski (1999) for a discussion).