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STOCHASTIC INTEREST RATE MODELS

Typical Setup

Take as given the market price process, $S(t)$, of some underlying asset.

$S(t)$ = price, at t, per unit of underlying asset

Consider a fixed **financial derivative**, e.g. a European call option.

Main Problem: Find the arbitrage free price of the derivative.

BASIC TOOLS

We Need:

1. Mathematical model for the underlying price process. (The Black-Scholes model)

2. Mathematical techniques to handle the price dynamics. (The Itô calculus.)

STOCHASTIC PROCESSES

- Discrete vs continuous time stochastic processes:
  - Discrete – the variable value can change only at certain fixed points in time
  - Continuous – changes can take place at any time

- Continuous vs discrete variables:
  - Discrete – only certain values are possible
  - Continuous – can take any value within a certain range

- Continuous-variable, continuous-time – variables can assume any value and changes can occur at any time.

Stochastic Processes

Stochastic variable

Choosing a **number** at random

Stochastic process

choosing a **curve** (trajectory) at random.

STOCHASTIC PROCESSES

- Continuous-variable, continuous-time stochastic processes are key to understanding the pricing of options and other derivatives.
- However, in practice, most asset prices do not follow continuous-variable, continuous-time stochastic processes.
- For instance, stock prices are restricted to discrete values (e.g. multiples of a cent) and changes can be observed only when the markets are open.
- Nonetheless, continuous-variable, continuous-time stochastic processes are useful for many valuation purposes.

Stochastic Processes

- We model the stock price $S(t)$ as a **stochastic process**, i.e. it **evolves randomly over time**.

- We model S as a **Markov process**, i.e. in order to predict the future only the present value is of interest. All past information is already incorporated into today's stock prices.

(Market efficiency – weak form)

Stochastic Processes

- If the weak form of market efficiency were not true, market analysts could make above-average returns by interpreting the past behavior of asset prices.

- It is the competition that tends to ensure that weak-form market efficiency holds.

Notation

$$X(t) = \text{any random process},$$
$$dt = \text{small time step},$$
$$dX(t) = X(t+dt) - X(t)$$

- dX is called the **increment** of X over the interval $[t, t+dt]$.
- For any fixed interval $[t, t+dt]$, the increment dX is a stochastic variable.
- If the increments $dX(s)$ and $dX(t)$, over the disjoint intervals $[s, s+ds]$ and $[t, t+dt]$ are independent, then we say that X has **independent increments**.
- If every increment has a normal distribution we say that X is a **normal**, or **Gaussian** process.

WIENER PROCESS

A stochastic process W is called a **Wiener process (or the continuous random walk)** if it has the following properties:

- The increments are normally distributed:

  For $s < t$:

$$W(t) - W(s) \sim N[0, \sqrt{t-s}]$$

$$E[W(t) - W(s)] = 0, \quad Var[W(t) - W(s)] = t - s$$

- W has independent increments.
- $W(0) = 0$
- W has continuous trajectories.

Theorem:

A Wiener trajectory is, with probability one, **nowhere differentiable**.

GENERALIZED WIENER PROCESS

A stochastic process X is called a Wiener process with **drift** μ and **diffusion coefficient** σ if it has the following dynamics

$$\begin{aligned} dX &= \mu dt + \sigma dW, \\ X(0) &= x_0 \end{aligned}$$

where x_0, μ and σ are constants.

Summing all increments over the interval $[0, t]$ gives us $X(t) - x_0 = \mu t + \sigma W(t)$

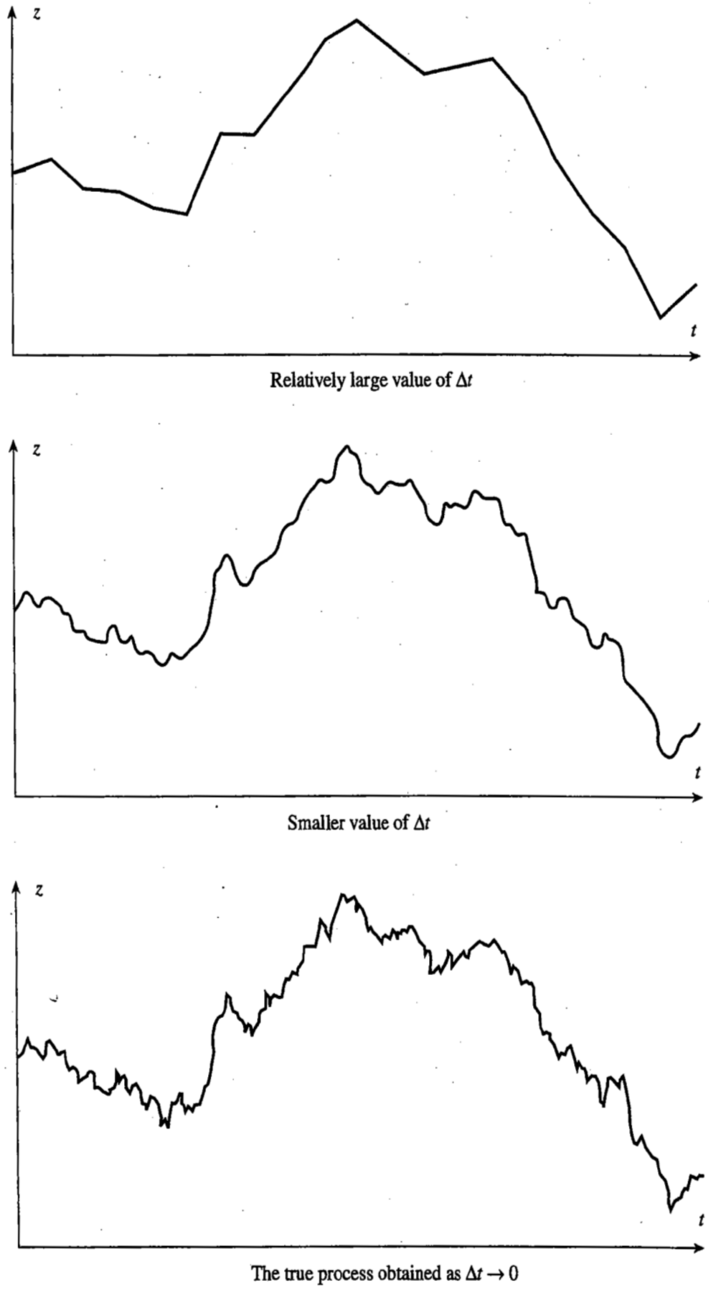
The distribution of X is thus given by

$$X(t) \sim N[x_0 + \mu t,\ \sigma\sqrt{t}]$$

- Therefore , uncertainty is proportional to the square root of time.
- The average increases are proportional to time (if there is no drift, the average doesn't change).

WIENER PROCESS

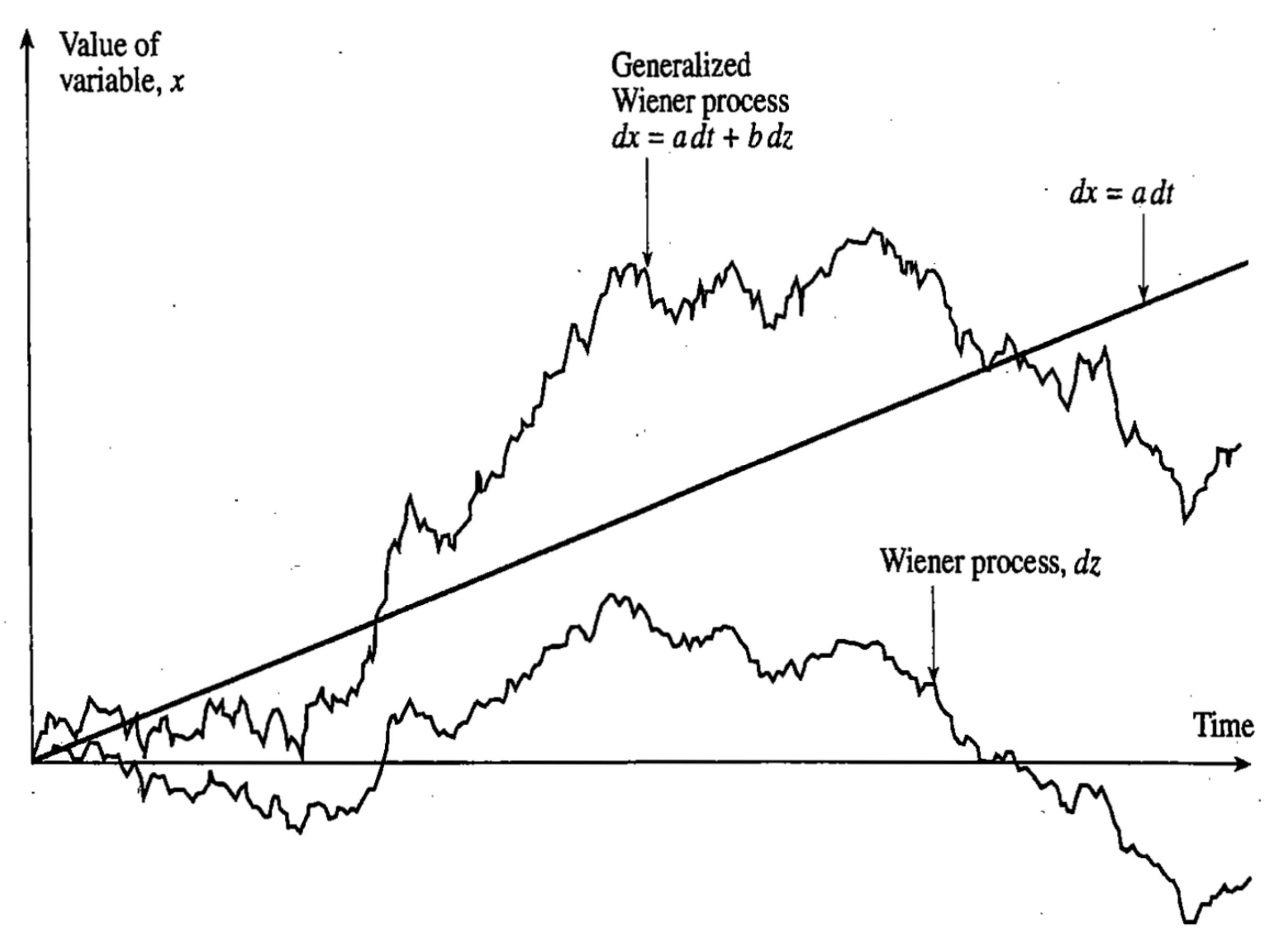
Wiener processes for different magnitudes of change in time:



When Δt -> 0, the path becomes much more irregular, as the size of the movement in the variable in time Δt is proportional to the $\sqrt{\Delta t}$. When Δt is small, $\sqrt{\Delta t}$ is much larger than Δt.

Source: Hull, John (2009), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 7th Edition

GENERALIZED WIENER PROCESS



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GENERALIZED WIENER PROCESS

- Generalized Wiener process with initial value of 50, annual drift and variance of 20 and 900, respectively:

t	0	0.5	1
μ	20	60 (=50+20*0,5 = $x_0+\mu*dt$)	70 (=50+20)
σ^2	900	21.21 (=30*sqrt(0,5)=σ*$sqrt(t)$)	30 (=30*sqrt(1))

$$\begin{aligned} dX &= \mu dt + \sigma dW, \\ X(0) &= x_0 \end{aligned}$$

where x_0, μ and σ are constants.

Itô Process

- Definition: Generalized Wiener process with average and standard-deviation as functions of the underlying variable and time (instead of constant along time):

$$dX = \mu(t, X_t)dt + \sigma(t, X_t)dW,$$

- Please recall that

  - The increments are normally distributed:
    For $s < t$:

$$W(t) - W(s) \sim N[0, \sqrt{t-s}]$$

$$E[W(t)-W(s)] = 0, \quad Var[W(t)-W(s)] = t-s$$

- For small time intervals, we may assume that the average and the standard-deviation don't change:

$$\Delta X = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

ITÔ PROCESS

- It may be tempting to assume that a stock price follows a generalized Wiener process (constant drift and variance).
- However, this assumption is not valid, having in mind that investors require or expect a given level of returns (as a % variation) regardless the price level, i.e. for higher prices, expected changes will be higher too.

- One can replace the assumption of constant expected drift by the assumption of constant expected returns (i.e. constant expected drift divided by the stock price).

ITÔ PROCESS

- If S is the stock price at time t => expected drift rate in S must be μS (being μ constant, corresponding to the expected rate of return on the stock, expressed in decimal form).
- In a short interval of time Δ_t, the expected increase in S is $\mu S \Delta_t$, i.e the expected rate of return on the stock, times the stock price, times the time interval:

$$\Delta S = \mu S \Delta t$$

- If **Δt -> 0 =>**

$$dS = \mu S dt \Leftrightarrow \frac{dS}{S} = \mu dt$$

- This corresponds to the price of an asset following a continuously compounding process: $S_T = S_0 e^{\mu T}$

GEOMETRIC BROWNIAN MOTION

- Given that asset prices actually exhibit volatility, a reasonable assumption is that the variability of the percentage return in a short period of time Δ_t is the same regardless the stock price.
- This means that an investor is as uncertain about his return when the stock price is high or low.
- Accordingly, the standard deviation of the change in a short period of time must be proportional to the stock price, as the standard deviation for the percentual change is constant – **Geometric Brownian Motion**:

$$dS = \mu S dt + \sigma S dz \Leftrightarrow$$

$$\frac{dS}{S} = \mu dt + \sigma dz$$

ITÔ'S LEMMA

- An option price is a function of the underlying asset's price and time.
- Therefore, it is important to understand the behavior of functions of stochastic variables.
- An important result was discovered by K. Itô in 1951 and is known as **Itô's lemma**.
- Assuming that a variable x follows a Itô process:

$$dx = a(x, t)\, dt + b(x, t)\, dz$$

where dz is a Wiener process and a and b are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2. Itô's lemma shows that a function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\, dz$$

ITÔ'S LEMMA

- Thus, G (the derivatives price) also follows an Itô process with a drift rate

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

$$dX = \mu(t, X_t)dt + \sigma(t, X_t)dW,$$

and a variance rate of

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2$$

- Assuming that the stock price follows a Geometric Brownian Motion, with constant μ and σ:

$$dx = a(x, t)\,dt + b(x, t)\,dz$$

$$dS = \mu S dt + \sigma S dz$$

- From Ito's Lemma it follows that

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S\,dz$$

ITÔ'S LEMMA

- Therefore, both *S* and *G* are affected by the same volatility source – *dz*.

- This is in line with the Black-Scholes option pricing formula, as G (the option price) is determined by the instantaneous volatility of the returns of the underlying asset price.

ITÔ'S LEMMA

How can Itô's Lemma be considered as a natural extension of simpler results?

- Assuming a continuous and differentiable function *G* of a variable *x*.
- If Δx is a small change in x and ΔG is the resulting change in *G*, one gets

$$\Delta G \approx \frac{dG}{dx}\Delta x$$

- Thus, ΔG is approximately equal to the rate of change of *G* with respect to *x*, multiplied by Δx.

ITÔ'S LEMMA

- If more precision is required, a Taylor series expansion of ΔG can be used:

$$\Delta G = \frac{dG}{dx}\Delta x + \tfrac{1}{2}\frac{d^2G}{dx^2}\Delta x^2 + \tfrac{1}{6}\frac{d^3G}{dx^3}\Delta x^3 + \cdots$$

- For a continuous and differentiable function G of 2 variables x and y, one obtains

$$\Delta G \approx \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y$$

- and the Taylor series expansion becomes

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial y}\Delta x\,\Delta y + \tfrac{1}{2}\frac{\partial^2 G}{\partial y^2}\Delta y^2 + \cdots$$

Itô's Lemma

- When Δx and Δy tends to zero, one gets

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy$$

- Extending this version in order to cover functions of variables following Itô processes – $dx = a(x, t)\, dt + b(x, t)\, dz$ -

- one obtains (basically replacing *y* by *t*):

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \tfrac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta x^2 + \frac{\partial^2 G}{\partial x\, \partial t} \Delta x\, \Delta t + \tfrac{1}{2} \frac{\partial^2 G}{\partial t^2} \Delta t^2 + \cdots$$

ITÔ'S LEMMA

- Discretizing

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

- one gets:

$$\Delta x = a(x,t)\,\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$$

- Or, by dropping the arguments,

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$$

- From this equation we get

$$\Delta x^2 = b^2\epsilon^2\Delta t + \text{terms of higher order in } \Delta t$$

Itô's Lemma

- This shows that in equation

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial t}\Delta x\,\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \cdots$$

- there are terms in Δx^2 that depend on t.

ITÔ'S LEMMA

- Given that the variance of a standardized normal distribution =1:

$$E(\epsilon^2) - [E(\epsilon)]^2 = 1$$

- As $E(\epsilon) = 0$, => $E(\epsilon^2) = 1$ => $\epsilon^2 \Delta t$ is non-stochastic and equal to its expected value Δt.

- Consequently, Δx^2 becomes non-stochastic and equal to $b^2 dt$.

Itô's Lemma

- Therefore, taking limits as Δt and Δx tend to zero in equation

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial t}\Delta x\,\Delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \cdots$$

- one gets:

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 dt$$

- If we consider that $dx = a(x,t)\,dt + b(x,t)\,dz$, replacing *dx* in the former expression by the latter, one obtains

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$

PROBABILITY DISTRIBUTION

- From the stochastic process of the rate of returns,

$$\frac{dS}{S} = \mu dt + \sigma dz$$

- Its distribution gets

$$\frac{\Delta S}{S} \sim \phi(\mu \Delta t, \sigma^2 \Delta t)$$

- Being $G = \ln S$, since $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, $\frac{\partial G}{\partial t} = 0$, it follows from the Itô's lemma that

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dz$$

PROBABILITY DISTRIBUTION

Since μ and σ are constant, this equation indicates that $G = \ln S$ follows a generalized Wiener process. It has constant drift rate $\mu - \sigma^2/2$ and constant variance rate σ^2. The change in $\ln S$ between time 0 and some future time T is therefore normally distributed, with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$. This means that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

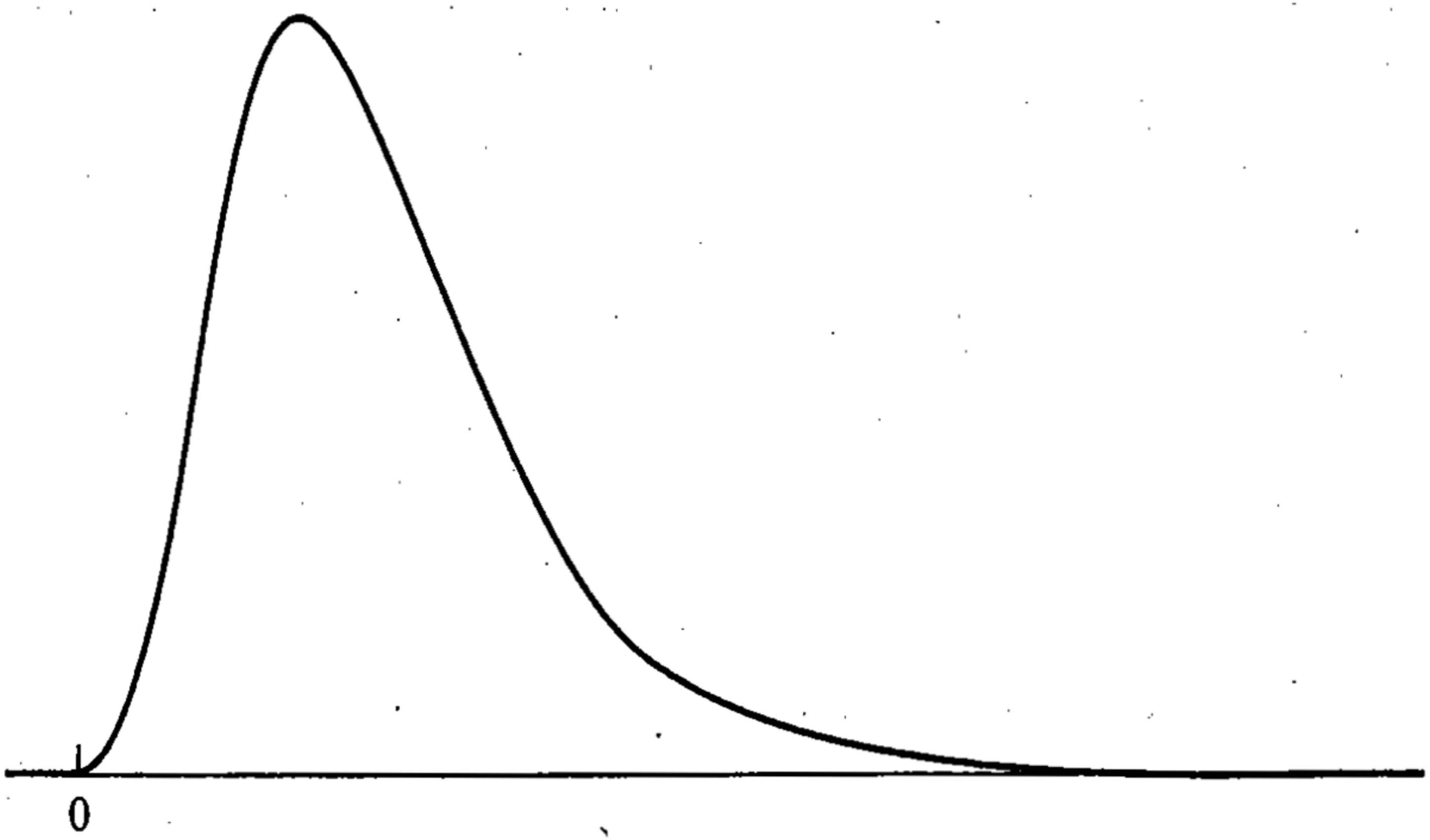
or

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

- This equation shows that $\ln S_T$ is normally distributed (and S_T has a log normal distribution), with a standard deviation $\sigma\sqrt{T}$ that is proportional to the square root of time.

PROBABILITY DISTRIBUTION

Figure 13.1 Lognormal distribution.



Source: Hull, John (2009), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 7th Edition

MARTINGALES

Definition: A **martingale** is a zero drift stochastic process:

$$dX = \sigma dz \qquad \text{(where } dz \text{ is a Wiener process)}$$

⇩

The best estimate for the future price is the current one:

$$E(X_T) = X_0$$

Risk-Neutral Valuation Result

$$\Pi\left[t; X\right] = e^{-r(T-t)} E^{Q}_{t,s}\left[X\right]$$

X is the payoff at T

Q-dynamics:

$$\begin{cases} dS = rSdt + \sigma SdW, \\ dB = rBdt. \end{cases}$$

Assuming constant interest rates

- Price = Expected discounted value of future payments.
- The expectation shall **not** be taken under the "objective" probability measure P, but under the "risk adjusted" measure ("martingale measure") Q.

Reasoning

- When we compute prices, we can compute **as if** we live in a risk neutral world.
- This does **not** mean that we live (or think that we live) in a risk neutral world.
- The valuation formulas are therefore called "preference free valuation formulas".

Example: The Black-Scholes Formula

$$\Pi\left[t;X\right] = e^{-r(T-t)}E_{t,s}^{Q}\left[X\right]$$

$$X = \max[S_T - K, 0]$$

$$\Pi\left[t;X\right] = sN\left[d_1\right] - e^{-r(T-t)}KN\left[d_2\right]$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

$N[\cdot]$=cdf for $N(0,1)$-distribution.

Market Completeness

- We **assumed** that the derivative was traded. How do we price OTC products?

- Suppose that we have sold a call option. Then we face financial risk, so how do we hedge against that risk?

All this has to do with **completeness**.

REACHABLE (OR HEDGEABLE) CLAIM

Definition:
We say that a T-claim X can be **replicated**, alternatively that it is **reachable** or **hedgeable**, if there exists a self financing portfolio h such that

$$V^h(T) = X, \quad P - a.s.$$

In this case we say that h is a **hedge** against X. Alternatively, h is called a **replicating** or **hedging** portfolio. If every contingent claim is reachable we say that the market is **complete**

Basic Idea: If X can be replicated by a portfolio h then the arbitrage free price for X is given by

$$\Pi[t; X] = V^h(t).$$

Classical Example:

The Black-Scholes Model