Probability Theory and Stochastic Processes

LIST 2

Measurable functions, Lebesgue integral

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

(1) Show that the sum of measurable functions is also a measurable function.

(2) Show that a function f is measurable iff f^+ and f^- are measurable. (*Hint:* $f^+ = \mathcal{X}_A f$ where $A = \{x \in \Omega \colon f(x) > 0\}$.)

(3) Let f be a measurable function. Show that
   (a) $|f|$ is also measurable.
   (b) $f^{-1}(\{a\}) = \{x : f(x) = a\}$ is a measurable set.

(4) Let $\mathcal{A} \subset \mathcal{F}$ be σ-algebras. Are the following propositions true? If not, write examples that contradict the statements.
   (a) If a function is $\mathcal{A}$-measurable, then it is also $\mathcal{F}$-measurable.
   (b) If a function is $\mathcal{F}$-measurable, then it is also $\mathcal{A}$-measurable.

(5) Prove the implications in the following sequence of propositions:

$$\text{uniform convergence} \Rightarrow \text{pointwise convergence} \Rightarrow$$

$$\Rightarrow \text{convergence a.e.} \Rightarrow \text{convergence in measure}$$

(6) Let μ be the counting measure. Consider $A = \{a_1, a_2, a_3\} \in \mathcal{F}$ and a measurable function f.
   (a) Is $\mathcal{X}_A f$ a simple function?
   (b) Compute $\int_A f \, d\mu$.

(7) (Markov inequality) Consider a measurable function $f \geq 0$. Show that for any $\lambda > 0$ we have

$$\mu\left(\{x \in \Omega \colon f(x) \geq \lambda\}\right) \leq \frac{1}{\lambda} \int f \, d\mu.$$

(8) Given measures μ_1, μ_2 and $c_1, c_2 \geq 0$, let $\mu = c_1\mu_1 + c_2\mu_2$. Take any function f which is simultaneously μ_1-integrable and μ_2-integrable. Show that f is also μ-integrable and that

$$\int f \, d\mu = c_1 \int f \, d\mu_1 + c_2 \int f \, d\mu_2.$$

Hint: First prove it for simple functions and then use the monotone convergence theorem.

(9) Let $\mathcal{A} \subset \mathcal{F}$ a σ-subalgebra, f, g are $\mathcal{F}$-measurable functions and h is $\mathcal{A}$-measurable. Are the following propositions true? If not, write examples that contradict the statements.
(a) If $\int_B f\,d\mu = \int_B g\,d\mu$ for every $B \in \mathcal{F}$, then $f = g$ a.e.
(b) If $\int_A f\,d\mu = \int_A h\,d\mu$ for every $A \in \mathcal{A}$, then $f = h$ a.e.

(10) Use the dominated convergence theorem to determine the limits (where δ_a stands for the Dirac delta measure at a):
(a) $\lim_{n\to+\infty} \int_0^\pi \frac{\sqrt[n]{x}}{1+x^2}\,d\delta_0(x)$
(b) $\lim_{n\to+\infty} \int_0^\pi \frac{\sqrt[n]{x}}{1+x^2}\,dx$
(c) $\lim_{n\to+\infty} \int_{-\infty}^{+\infty} e^{-|x|}\cos^n(x)\,dx$
(d) $\lim_{n\to+\infty} \int_0^{+\infty} \frac{r^n}{1+r^{n+2}}\,d\delta_1(r)$
(e) $\lim_{n\to+\infty} \int_0^{+\infty} \frac{r^n}{1+r^{n+2}}\,dr$