Probability Theory and Stochastic Processes

LIST 4

Limit theorems and conditional expectation

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(1) If X and Y are independent random variables, show that for any measurable functions f and g:
   (a) $f(X)$ and $g(Y)$ are independent.
   (b) $E(f(X)g(Y)) = E(f(X))\,E(g(Y))$ if $E(|f(X)|), E(|g(Y)|) < +\infty$.

(2) * Show that the weak law of large numbers does not hold for the Cauchy distribution.

(3) Given a sequence X_n of iid random variables with uniform distribution on $[0,1]$, determine

$$\lim_{n\to+\infty} \sqrt[n]{X_1 \dots X_n}$$

with probability 1 (i.e. almost surely).

(4) * Given a sequence X_n of random variables such that $P(X_i = 2^i) = 2^{-i}$, $P(X_i = 0) = 1 - 2^{-i}$, $i \geq 1$, determine

$$\lim_{n\to+\infty} \frac{X_1 + \cdots + X_n}{n}$$

with probability 1. *Hint:* Use the Borel-Cantelli lemma. Notice that the strong law of large numbers does not hold.

(5) Let $C \subset \Omega$, the σ-algebra

$$\mathcal{F} = \{\emptyset, \Omega, C, C^c\}$$

and the probability measures on $\mathcal{F}$ given by

$$\mu(C) = \frac{1}{2} \quad \text{and} \quad \lambda(C) = \frac{1}{4}.$$

Consider also the trivial σ-algebra $\mathcal{A} = \{\emptyset, \Omega\} \subset \mathcal{F}$.
   (a) Show that $\lambda << \mu$.
   (b) Compute $f = \frac{d\lambda}{d\mu}$. Is it $\mathcal{F}$-measurable? Is it $\mathcal{A}$-measurable?
   (c) Compute $g = \frac{d\lambda}{d\mu}|_{\mathcal{A}}$. Is it $\mathcal{A}$-measurable?

(d) Prove that $g = E(f|\mathcal{A})$, i.e.

$$\int_A g\,d\mu = \int_A f\,d\mu, \quad A \in \mathcal{A}.$$

(6) Let $\Omega = [0,1[$, $\mathcal{F} = \mathcal{B}([0,1[)$ and $P = m$ where m is the Lebesgue measure on $[0,1[$. Consider the random variables $X(\omega) = \omega$ and

$$Y(\omega) = \begin{cases} 2\omega, & 0 \le \omega < \frac{1}{2} \\ 2\omega - 1, & \frac{1}{2} \le \omega < 1. \end{cases}$$

(a) Find $\sigma(Y)$.

(b) By the knowledge that $E(X|Y)$ is $\sigma(Y)$-measurable, show that

$$E(X|Y)(\omega) = E(X|Y)(\omega + 1/2), \quad 0 \le \omega < 1/2.$$

(c) Reduce the problem of determining $E(X|Y)$ on $[0,1[$ to finding the solution of

$$\int_A E(X|Y)\,dm = \frac{1}{2}\int_{A\cup(A+1/2)} X\,dm, \quad A \in \mathcal{B}([0,1/2[),$$

and compute $E(X|Y)$.

(7) Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ be σ-subalgebras of $\mathcal{F}$ and X an integrable function. Show that

$$E(X|\mathcal{F}_1) = E(E(X|\mathcal{F}_2)|\mathcal{F}_1) \text{ a.e.}$$

(8) Let $B \in \mathcal{F}$ with $P(B) > 0$. Compute:

(a) $\sigma(\mathcal{X}_B)$.

(b) $E(X|\mathcal{X}_B)$ for any random variable X.

(c) $P(A|\mathcal{X}_B)$ where $A \in \mathcal{F}$.