Mathematical Finance

Lisbon, December 13, 2016

All notation must be clear. Arguments should be complete.

1. Consider a standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

   and a simple T-claim X of the form

$$X = \Phi(S_T)$$

   Prove that the contract Φ can be replicated by a self financing portfolio based on S and B.

2. Consider a model for two countries. We then have a domestic market and a foreign market. The domestic and foreign interest rates, r_d and r_f, are assumed to be given real numbers. Consequently, the domestic and foreign savings accounts satisfy

$$B_t^d = e^{r_d t}, \qquad B_t^f = e^{r_f t},$$

   where B^d and B^f are denominated in units of domestic and foreign currency, respectively. The exchange rate process X, which is used to convert foreign payoffs into domestic currency, is modelled by the following stochastic differential equation under the objective measure P

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

   where μ and σ are assumed to be constants and W is a P-Wiener process.

   A *domestic martingale measure*, Q^d, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of domestic currency and discounted using the domestic risk-free rate, martingales. We assume that if you buy the foreign currency this is immediately invested in a foreign bank account. All markets are assumed to be frictionless.

A *foreign martingale measure*, Q^f, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of foreign currency and discounted using the foreign risk-free rate, martingales.

Your task is to find the Girsanov transformation between Q^d and Q^f, and to derive the Q^d-dynamics of X.

3. Let the stock prices S^1 and S^2 be given as the solutions to the following system of SDE:s.

$$
\begin{aligned}
dS_t^1 &= \alpha S_t^1 dt + \delta S_t^1 dW_t^1, \quad S_0^1 = s_1, \\
dS_t^2 &= \beta S_t^2 dt + \gamma S_t^2 dW_t^2, \quad S_0^2 = s_2,
\end{aligned}
$$

The Wiener processes W^1 and W^2 are assumed to be independent. The parameters α, δ, γ, β are assumed to be known and constant. Your task is to price a **maximum option**. This T-claim is defined by

$$X = \max\left[S_T^1, S_T^2\right]$$

The pricing function for a European call option in the Black-Scholes model is assumed to be known, and is denoted by $c(s, t; K, \sigma, r, T)$ where σ is the volatility, K is the strike price and r is the short rate. You are allowed to express your answer in terms of this function, with properly derived values for K, σ and r.

4. Consider the problem to maximize

$$E\left[\int_0^T f(X_s, u_s)ds + F(X_T)\right]$$

given the controlled diffusion

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t$$

All processes are scalar and there are no constraints on the control u. Derive the Hamilton-Jacobi-Bellman equation for this problem.