Mathematical Finance

Lisbon, January 2016

All notation must be clear. Arguments should be complete.

1. Consider a standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

and a simple T-claim X of the form

$$X = \Phi(S_T)$$

Derive the Black-Scholes PDE, including the boundary condition, for the pricing function $F(t, s)$.

2. Consider a Black-Scholes model (under P) with constant dividend yield q.

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dD_t &= q S_t dt, \\ dB_t &= r B_t dt. \end{aligned}$$

and a simple T-claim X of the form

$$X = \Phi(S_T)$$

(a) Derive the relevant risk neutral pricing formula for the pricing function $F^q(t, s)$.

(b) Show how F^q is related to the corresponding pricing function for the case of zero dividend yield. In other words: Derive the relationship between F^q and F^0.

You are allowed, without proof, to use the dynamics of a self financing portfolio in the case of dividends, as well as the fact that the normalized gain process is a martingale under Q.

3. Consider a model for two countries. We then have a domestic market and a foreign market. The domestic and foreign interest rates, r_d and r_f, are assumed to be given real numbers. Consequently, the domestic and foreign savings accounts satisfy

$$B_t^d = e^{r_d t}, \qquad B_t^f = e^{r_f t},$$

where B^d and B^f are denominated in units of domestic and foreign currency, respectively. The exchange rate process X, which is used to convert foreign payoffs into domestic currency, is modelled by the following stochastic differential equation under the objective measure P

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

where μ and σ are assumed to be constants and W is a P-Wiener process.

A *domestic martingale measure*, Q^d, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of domestic currency and discounted using the domestic risk-free rate, martingales. We assume that if you buy the foreign currency this is immediately invested in a foreign bank account. All markets are assumed to be frictionless.

A *foreign martingale measure*, Q^f, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of foreign currency and discounted using the foreign risk-free rate, martingales.

Your task is to find the Girsanov transformation between Q^d and Q^f, and to derive the Q^d-dynamics of X.

4. Consider a standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

and let X_t denote the value process of a self financing portfolio. Solve the problem of maximizing log utility of terminal wealth, i.e. you want to maximize

$$E^P\left[\ln\left(X_T\right)\right]$$