LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Master in Actuarial Science
Rate Making and Experience Rating

Exam 1, 06/01/2017
Time allowed: 2:30

Instructions:

1. This paper contains 5 groups of questions and comprises 3 pages including the title page;
2. Enter all requested details on the cover sheet;
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so;
4. Number the pages of the paper where you are going to write your answers;
5. Attempt all questions;
6. Begin your answer to each of the 5 question groups on a new page;
7. Marks are shown in brackets. Total marks: 200;
8. Show calculations where appropriate;
9. An approved calculator may be used. No mobile phones or other communication devices are permitted;
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Consider a certain insurance portfolio where risks can be of three different types, say, 1, 2, and 3. Per year each risk can produce at most 2 claims, and each claim amount are of a fixed/constant quantity. Probabilities for each type of risk are shown in the table below. Suppose that the composition of each risk type in the population is 70%, 25% and 5%, respectively for types 1, 2 and 3.

Type \	0	1	2
1	0.50	0.35	0.15
2	0.40	0.35	0.25
3	0.25	0.50	0.25

Consider that the usual hypothesis of the Bayesian credibility theory are fulfilled.

Suppose that two risks are taken at random and independently, with replacement, from the portfolio. Let X be the the total of claims from the two risks drawn, its p.f. is denoted as f_X, and let the random variable Θ represent the risk type, $\Theta = 1, 2, 3$, with p.f. $\pi(\theta)$. **[80]**

(a) For each type of risk, calculate the conditional probaility $f_{X|\theta}(x|\theta = i)$, $i = 1, 2, 3$. (15)

(b) Determine the (unconditional) mean and variance of X. (15)

(c) Each double drawing is done with replacement and X is observed. From now onwards consider the following: In a first drawing a total of 3 claims was observed and in a second a total of 1 claim, i.e, we observed $(X_1 = 3, X_2 = 1)$.
Calculate the probability of event $(X_1 = 3, X_2 = 1)$. (7.5)

(d) Calculate the posterior distribution $\pi_{\Theta|(x_1,x_2)}(\theta|x_1 = 3, x_2 = 1)$. (12.5)

(e) Calculate the Bayesian premium (for the "next outcome"). (10)

(f) Determine Bühlmann's credibility premium, P_c. (15)

(g) Should the case be considered as "Exact Credibility"? Explain shortly. (5)

2. Consider a portfolio where a certain risk X, in each year, can produce at most a claim, of a certain fixed amount, with probability θ. Furthermore, θ can be considered as an outcome of a random variable Θ following a Beta distibution with parameters $\alpha = 0.3$ and $\beta = 0.7$.

In the last five years the risk as produced an average number of claims of 2/5. **[25]**

(a) Write and explain shortly all the model assumptions that you need to assume in order to estimate the next year premium for the risk. (5)

(b) Calculate the Bayesian premium. Show and explain clearly all the steps taken. (20)

3. Suppose you have n observations for a certain risk, X_j, $j = 1, 2, ..., n$, in a portfolio. Consider Bühlmann's H_1 and H_2. The Credibility (pure) Premium of the risk for year $n + 1$ is defined as the linear estimnator $\widetilde{\mu_{n+1}} = \alpha_0 + \sum_{j=1}^{n} \alpha_j X_j$, where α_i, $i = 0, 1, 2, \ldots, n$, are such that:

$$\min Q = E\left\{\left[\mu_{n+1}(\theta) - \widetilde{\mu_{n+1}}\right]^2\right\} .$$

Show that the solution of the problem can be taken from equations:

$$\begin{aligned} E(X_{n+1}) &= E\left(\widetilde{\mu_{n+1}}(\theta)\right); \\ E[X_i X_{n+1}] &= \tilde{a}_0 E[X_i] + \sum_{j=1}^{n} \tilde{\alpha}_j E\left[X_i X_j\right], \ i = 1, ..., n. \end{aligned}$$

[15]

4. A certain insurer is considering a *bonus-malus* system (BMS) based on the individual's annual claims record to rate each individual risk in a given motor insurance portfolio. **[50]**

(a) Suppose you are building a system where, apart from the entry class with index premium 100, you consider two classes of bonus with 20% and 30% discount, reachable if you have two or three years in a row with no claims at all, respectively. Apart from that you consider a penalty class with an overcharge of 40% in the case of a claim in a year.
Build a Markovian system by setting either a table with the rules or the rules matrix.

(b) For the system in the item above, argue shortly about the long term behaviour.

(c) State how would you compare two different BMS's. Give two examples.

(d) *The intention of a BMS is to approximate a (pure) premium as close as possible to the annual expected risk behaviour. Classical BMS's assume that changes on the number of claims occurred should be compensated by a change in the expected value of the risk accordingly.* Discuss brief and appropriately the statement, considering that you are facing a BMS based on number of claims, either Poisson(λ) or mixed Poisson distributed.

5. A working party is modelling a tariff for a given large motor insurance portfolio.

The study group is proposing a new tariff for an existing portfolio evaluating a wide variety of commonly used risk factors that might have impact in both the claim frequency and the claim size means. Each risk factor may be divided into a short number of different levels. **[30]**

(a) The insurer uses a *bonus-malus* system to charge each individual next year premium according to his current year claims. Comment the statement: *The bonus or malus levels should not be used as a risk factor for tariffication purpose.*

(b) For the estimation of the pure premium, explain briefly how, and why, we could use GLM's to model the pure premium altogether or we could model claim counts and claim sizes separately.

(c) Since the pure premium is given by the product between the expect claim counts and the claim size expectation, it's obvious that the study group should build the tariff based on a multiplicative model. Comment shortly.

(d) Suppose that when modelling the "expected claim size units" (N) the group came out with the following final model:

$$\begin{aligned}\ln N \quad = \quad & -2.62863 + .27076x_{2,2} + .34211x_{2,3} + .20249x_{4,2} - 0.12602x_{5,2} + .17053x_{7,3} + .23968x_{7,4} \\ & + .21130x_{8,1} + .20062x_{9,1} + .77360x_{9,2} + .28878x_{10,1} - 0.22840x_{12,3} - 0 : 50593x_{12,4} \,,\end{aligned}$$

where $x_{i,j}$ corresponds for risk factor i and level j.
Write the base premium for a risk with an exposure of 120 days (year: 365 days).

Solutions:

1. (a) $f_{X|\Theta}(x|\theta = i)$:

$x\backslash i$	1	2	3
0	0.25	0.16	0.0625
1	0.35	0.28	0.25
2	0.2725	0.3225	0.375
3	0.105	0.175	0.25
4	0, 0225	0.0625	0.0625
	1	1	1

(b)

$$\begin{aligned}
\mu(1,2,3) &= (1.3; 1.7; 2.0) \\
\mu &= 0.7(1.3) + 0.25(1.7) + 0.05(2) = 1.435 \\
\nu(1,2,3) &= (1.055; 1.255; 1.0) \\
\nu &= 1.102\,25 \\
a &= 0.7(1.3^2) + 0.25(1.7^2) + 0.05(2^2) - 1.435^2 = 0.0462\,75 \\
V[X] &= a + \nu = 1.148\,525\,.
\end{aligned}$$

(c) $Pr(X_1 = 3, X_2 = 1) = \sum_{i=1}^{3} Pr(X_1 = 3)Pr(X_2 = 1)\pi(i) = 0.0411.$

(d)

$$\pi\left(i|(X_1 = 3, X_2 = 1)\right) = \frac{Pr(X_1 = 3, X_2 = 1|\theta = i)\pi(i)}{Pr(X_1 = 3, X_2 = 1)}\,,$$

i	$\pi(i)$	$\pi\left(i|(X_1 = 3, X_2 = 1)\right)$
1	0.70	0.625 91
2	0.25	0.298 05
3	0.05	0.076 04
	1	1

(e) Let X_3 be the next outcome.

$$E(X_3|X_1 = 3, X_2 = 1) = E[\mu(\theta)|X_1 = 3, X_2 = 1] = 1.3(0.625\,91) + 1.7(0.298\,05) + 2(0.076\,04) = 1.472\,44$$

(f) $P_c = z\bar{x} + (1 - z)\mu = 1.478\,765$, with $n = 2$, $z = 2/(2 + k)$, $k = \nu/a = 23.819\,557$.

(g) No exact credibility:

$$\mu = 1.435 < \widetilde{\mu}(\theta|3, 1) = 1.472\,44 < \widetilde{\mu}(\theta) = 1.478\,765 < \bar{x} = 2\,.$$

2.

(a) i. H_1: Given θ, $X_1, X_2, \ldots X_{n+1} \overset{iid}{\frown} f(x|\theta)$. Explain...
Also, θ is an outcome of a r.v. $\Theta \frown \pi(\theta)$. Explain...

ii. H_2: Risks $(X_1, \Theta_1), \ldots$ are independent and $\Theta_i \overset{d}{=} \Theta$.

(b) $X|\theta \frown B(1, \theta)$, $\Theta \frown Beta(0.3, 0.7)$, $\mu(\theta) = \theta$. These are conjugate distributions so, the posterior is of the same family and $\pi(\theta|\bar{x}) = Beta(n\bar{x} + \alpha, \beta + n - n\bar{x})$. Then

$$E[\mu(\theta)|\underline{x}] = E[\theta|\underline{x}] = \frac{n\bar{x} + \alpha}{n + \alpha + \beta} = \frac{2.3}{6}\,.$$

3. If you take the derivative with respect to α_0 you get immediately the 1st equation. If you take the derivative with respect to α_i, $i = 1, \ldots, n$, we get

$$E[\mu(\theta)X_i] = E[\widetilde{\mu}(\theta)X_i]\,.$$

Left hand side is $E[E[\mu(\theta)X_i|\Theta]] = E[X_{n+1}X_i]$. The righthand side is, directly, $\widetilde{\alpha}_0 E[X_i] + \sum_{j=1}^{n} \widetilde{\alpha}_j E\left[X_i, X_j\right]$, $i = 1, ..., n$.

4.

(a)

Level	%	New level after	
		0	1+ claims
1	70	1	6
2	80	1	6
3	100	2	6
4	100	3	6
5	140	2	6
6	140	5	6

(b) If you separete class of levels $\{3, 4\}$ and $\{1, 2, 5, 6\}$, draw a diagram, you see easily that a policy leaves "quickly" the 1st class to the 2nd, and will never return. Levels $\{3, 4\}$ are transient Markov states and the other levels are recurrent. In the long run policies will move among levels $\{1, 2, 5, 6\}$.

(c) In the long run, through evaluation measures, like RSAL (Relative Stationary Average Level) and Elasticity of the average premium, for instance.

(d) What the statement says is that policies should be penalized by the occurrence of claims, and on the other hand give discounts for "no claims". Classical systems don't say anything about severity of claims, so insurers act with information that it is incomplete and give way to behaviours like the search for bonus ("bonus hunger").

5.

(a) Not necessarily. The existence of a BMS may lead to a significant influence in the risk behaviour (like the *bonus hunger*)we must be cautious. The factor must be tested.

(b) Let S be aggregate claims, N and X be claim frequency and severity, respectively. Pure Premium is

$$E[S] = E[N]E[X] \, .$$

We can model the two expectations separately or on aggregate. If we separate we can we can "get" different factors or factors working in opposite ways.

(c) It has nothing to do with that.

(d)

$$\exp\{-2.62863\}\frac{120}{365} \sim 0.0237295 \, .$$