Master in Actuarial Science
Rate Making and Experience Rating

Exam 2, 01/02/2017
Time allowed: 2:30

Instructions:

1. This paper contains 4 groups of questions and comprises 3 pages including the title page;
2. Enter all requested details on the cover sheet;
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so;
4. Number the pages of the paper where you are going to write your answers;
5. Attempt all questions;
6. Begin your answer to each of the 4 question groups on a new page;
7. Marks are shown in brackets. Total marks: 200;
8. Show calculations where appropriate;
9. An approved calculator may be used. No mobile phones or other communication devices are permitted;
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Consider a certain insurance portfolio where each risk can be individualized by a specific characteristic. Denote the risk by the random variable X and the individual characteristic by parameter θ which quantifies the individual characteristic. For a given θ, let $X|\theta \frown \exp(\theta)$, with mean $1/\theta$. Let θ be the outcome of a random variable $\Theta \frown Gamma(4, 0.001)$, mean 0.004. That is

$$\begin{aligned} f_x(x|\theta) &= \theta e^{-\theta x}, \; x,\theta > 0\,, \\ \pi(\theta) &= \theta^3 e^{-1000\theta} 1000^4/6, \; \theta > 0\,. \end{aligned}$$

Assume that the usual hypothesis of the Bayesian credibility theory are fulfilled.

Consider that in the past a given risk had reported claim amounts of amounts 250, 800 and 450. Consider the prediction or estimation of a next claim X_4. **[85]**

(a) Insurers group risks in portfolios by similarities. However, in Credibility Theory we consider individualizing premia, giving space for risks to be different. Can you explain briefly existence of potential contradictions. (5)

(b) Determine the (unconditional) mean and variance of X. (15)

(c) Show that the joint density $f_X(250, 800, 450)$ and $f(250, 800, 450, x_4)$ are given by (15)

i.

$$f(250, 800, 450) = \frac{1,000^4}{6} \frac{720}{2,500^7}\,;$$

(10)

ii.

$$f(250, 800, 450, x_4) = \frac{1,000^4}{6} \frac{5040}{(2,500 + x_4)^8}\,.$$

(d) Show that the predictive density, for x_4, is a $Pareto(7; 2\,500)$ distribution. (10)

(e) Calculate the colective and the Bayesian premia. (10)

(f) Determine Bühlmann's credibility premium, P_c. (10)

(g) Calculate the posterior distribution. Comment briefly. (10)

2. Suppose you have n observations for a certain risk, X_j, $j = 1, 2, ..., n$, in a portfolio. The Credibility (pure) Premium of the risk for year $n+1$ is defined as the linear estimnator $\widetilde{\mu_{n+1}} = \alpha_0 + \sum_{j=1}^{n} \alpha_j X_j$, where α_i $i = 0, 1, 2, \ldots, n$ are such that:

$$\min Q = E\left\{ [\mu_{n+1}(\theta) - \widetilde{\mu_{n+1}}]^2 \right\}$$

Consider Bühlmann's credibility model and his assumptions. Bühlmann's credibility (pure) premium for the given risk, for the next year, is given by formula

$$P_c = z\bar{X} + (1-z)\mu,$$

where $z = n/(n + \upsilon/a)$, $\mu = E[\mu(\theta)]$, $\upsilon = E[\upsilon(\theta)]$, $a = V[\mu(\theta)]$; $\mu(\theta)$ and $\upsilon(\theta)$ are the risk mean and variance, respectively. **[35]**

(a) Explain briefly why we get the optimum values for α_j, $j = 1 \ldots, n$, all equal, so that we could minimize instead

$$\min Q^* = E\left\{ \left[\mu_{n+1}(\theta) - \left(\alpha_0 + \beta\bar{X}\right)\right]^2 \right\},$$

where $\bar{X}$ is the empirical mean of all observations and β is such that $\alpha_1 = \cdots = \alpha_n = \beta/n$. (10)

(b) Show that $E[P_c] = \mu$ and $Var[P_c] = z\,a$ (10)

(c) Explain the behaviour/variation of the credibility factor z as a function of υ or a, *ceteris paribus*. (15)

3. A certain insurer is considering a *bonus-malus* system (BMS) based on the individual's annual claims record to rate each individual risk in a given motor insurance portfolio. **[50]**

(a) Bonus systems are usually built under Markov chain framework where premia are computed and adjusted annually based on claim counts. Suppose you have a Markov 5-state system with two classes both with a 100 premium index, a sole bonus class and two penalty classes with the same premium.

Can you please give the main idea(s) behind the system shortly described?

Class number	Premium level %	New step after claims 0	1	2	3	4	5	6+
1	200	2	1	1	1	1	1	1
2	150	3	1	1	1	1	1	1
3	130	4	1	1	1	1	1	1
4	115	5	2	1	1	1	1	1
5	100	6	3	1	1	1	1	1
6	90	7	4	2	1	1	1	1
7	80	8	5	3	1	1	1	1
8	80	9	6	4	2	1	1	1
9	70	10	7	5	3	1	1	1
10	60	11	8	6	4	2	1	1
11	50	12	9	7	5	3	1	1
12	50	13	11	9	7	5	3	1
13	40	13	11	9	7	5	3	1

Table 1: Rules and premium percentages

(b) Consider a *bonus* system that evolves according to what shown in Table 1. Considering a Poisson($\lambda = 0.1$) distribution for the claim counts build the associated transition probability matrix.

(c) Refer to the system above and Table 1. Calculate the premium that an insured is expected to pay one year after he entered the system.

(d) Explain shortly on *pros and cons* of BMS's based on claim counts only.

4. A working party is modelling a tariff for a given large motor insurance portfolio. [**30**]

The study group is proposing a new tariff for an existing portfolio evaluating a wide variety of commonly used risk factors that might have impact in both the claim frequency and the claim size means. Each risk factor may be divided into a short number of different levels.

(a) Building a tariff with GLM's uses past portfoio data to estimate premia for the different rating classes considered. Explain what is meant by prior and posterior ratemaking.

(b) When modeling a tariff it is common to work with key ratios and relativities. Explain briefly why. You may use examples to illustrate.

(c) Consider the sentence: *If we consider modeling the pure premium using directly the aggregate claims instead of modelling separatley the expect claim counts and the claim size expectation, and subsequently multiplying, it's obvious that we don't need a multiplicative model.* Comment and explain what model you could use.

(d) Suppose that when modelling the "expected claim size units" (N) the group came out with the following final model:

$$\begin{aligned}\ln N \quad = \quad & -2.62863 + .27076x_{2,2} + .34211x_{2,3} + .20249x_{4,2} - 0.12602x_{5,2} + .17053x_{7,3} + .23968x_{7,4} \\ & + .21130x_{8,1} + .20062x_{9,1} + .77360x_{9,2} + .28878x_{10,1} - 0.22840x_{12,3} - 0 : 50593x_{12,4}\,,\end{aligned}$$

where $x_{i,j}$ corresponds for risk factor i and level$j$.

Calculate the decrease in the premium for a driver with characteristic $x_{5,2}$, relative to the base premium.

Solutions:

1. (a) Portfolios are comprised by homogeneous risks, they have the same parametric family of distributions. However, Credibility allows some heterogeneity represented by a parameter, say θ, which is an outcome of a random variable. The difference among risks is not major, not contradicting homogeneity...

   (b) $\mu(\theta) = 1/\theta$, $\nu(\theta) = 1/\theta^2$). Then $\mu = 1000/3$, $\nu = 1000^3/6$, and $a = 1000^3/6 - 1000^2/9$. Finally, $V[X] = \nu + a = 1000^3/3 - 1000^2/9$.

   (c) i.
   $$f(250, 800, 450) = \int_0^\infty f(250|\theta)f(800|\theta)f(450|\theta)\pi(\theta)d\theta = \frac{1,000^4}{6}\frac{720}{2,500^7}.$$

   ii. Similarly, we get $f(250, 800, 450, x_4) = \frac{1,000^4}{6}\frac{5040}{(2500+x_4)^8}$.

   (d)
   $$f(x_4|250, 800, 450) = \frac{f(x_4, 250, 800, 450)}{f(250, 800, 450)} = \frac{7(2500)^7}{(2500 + x_4)^8}.$$

   (e) Collective premium: $\mu = 1000/3$, already calculated. Bayesian premium: Mean of $Pareto(7, 1500)$: $2500/6$.

   (f) Buhlmann's premium equal to Bayesian premium $E[X_4|250, 800, 450]$, exact credibility.

   (g) Posterior is $Gamma(7, 2500^{-1})$, mean $7/2500$. Conjugate distributions, exact credibility.

2. (a) Since all the (past) means and variances of observations from the same risk are equal, given the minimizing criteria, there is no reason to give different importance (or weight) over past obs.

   (b) $E[P_c] = z\mu + (1 - z)\mu = \mu$, obvious.
   $$Var[P_c] = z^2 Var[\bar{X}] = z^2(\nu/n + a) = za$$

   (c) If $k = \nu/a$ is big, when compared to n, then credibility is low. This means that a is small when compaerd to ν, there is small variability among risks. So, portfolio is very homogeneous, the collective premium must be trusted.

   In the opposite way, high variability brings heterogeneity, you should take into more account the individually history. ...

3. (a) The system has two double levels with the same premium, this is to make the system Markovian, as the description indicates that an insured needs two years to get a discount.

   (b) Let $p_i = e^{-0.1}\frac{0.1^i}{i!}$ and d.f. P_i, $i = 0, 1, \ldots$ We get

$$P = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array}\begin{array}{c} \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \left(\begin{array}{ccccccccccccc}
1-p_0 & p_0 & & & & & & & & & & & \\
1-p_0 & & p_0 & & & & & & & & & & \\
1-p_0 & & & p_0 & & & & & & & & & \\
1-P_1 & p_1 & & & p_0 & & & & & & & & \\
1-P_1 & & p_1 & & & p_0 & & & & & & & \\
1-P_2 & p_2 & & p_1 & & & p_0 & & & & & & \\
1-P_2 & & p_2 & & p_1 & & & p_0 & & & & & \\
1-P_3 & p_3 & & p_2 & & p_1 & & & p_0 & & & & \\
1-P_3 & & p_3 & & p_2 & & p_1 & & & p_0 & & & \\
1-P_4 & p_4 & & p_3 & & p_2 & & p_1 & & & p_0 & & \\
1-P4 & & p_4 & & p_3 & & p_2 & & p_1 & & & p_0 & \\
1-P5 & & p_5 & & p_4 & & p_3 & & p_2 & & p_1 & & p_0 \\
1-P5 & & p_5 & & p_4 & & p_3 & & p_2 & & p_1 & & p_0
\end{array}\right) \end{array}$$

   (c) $PM = 200\,(1 - P_1) + 130p_1 + 90p_0$.

   (d) There is a lack of information on severities, it is certainly much simpler to implement such a system, however brings problems like the "bonus hunger". This may lower premia and increase costs of claims, by unreporting claims.

4. (a) Briefly, Prior ratemaking means rate the policies of the portfolio according to risks characteristics known, or estimated, in advance. Posterior rate means adapt premia according to each actual individual behaviour shown on the claims reporting.

   (b) Key ratios are relative random variables, relative to exposure, e.g. capital insured, time period. A relativity is a factor multiplying to a base index quantity given as a unit. The premium will be calculated relatively to a base premium.

   (c) It has nothing to do with that. A multiplicative model can be used irrrespective of that.

   (d)

$$\frac{e^{-2.62863}}{e^{-2.62863-0.12602}} = e^{0.12602}$$