Master in Actuarial Science

Models in Finance

12-01-2018

Time allowed: Two and a half hours (150 minutes)

Instructions:

1. This paper contains 6 questions and comprises 4 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used.

1. Consider that the share price of a non-dividend paying security is given by the stochastic process

$$S_t = S_0 \exp\left\{\frac{1}{2}\left(\mu - \sigma^2\right) t + \sigma B_t\right\}$$

where:

- B_t is a standard Brownian motion under the real world measure P,
- μ and σ are positive constants.

(a) Deduce the stochastic differential equation (SDE) satisfied by the discounted price process$\widetilde{S}_t$. (14)

(b) The process $\widetilde{S}_t$ is a martingale under real world measure P? And under the equivalent martingale measure or risk neutral measure Q, what would be the SDE satisfied by $\widetilde{S}_t$? (10)

(c) If $\mu = 0.16$, $\sigma = 0.15$, the continuously compounded risk-free interest rate is 5% p.a. and the time is measured in years, calculate the probability that the annual return of this security is larger than the interest rate and less than 40%. (12)

2. Consider the following discrete time model for the volatility of an equity index (model 1):

$$V_t = 0.07 + 0.15 V_{t-1} + 0.05 z_t, \quad t = 1, 2, 3, \ldots$$

where V_t is the volatility at time t years and $z_1, z_2, \ldots$are a sequence of i.i.d. random variables with normal distribution $N(0,1)$. The initial volatility is $V_0 = 0.12$ (that is, 12%). Consider also the continuous-time model defined by the SDE (model 2):

$$dV_t = -\lambda \left(V_t - \mu\right) dt + \beta dB_t,$$

where V_t is the volatility at time t years, B_t is the standard Brownian motion and the parameters λ, β and μ all take positive values.

(a) Determine the long-term distribution of V_t for model 1 and determine the numerical value of μ and a relationship between parameters λ and β in such a way that V_t has the same long-term mean and variance under both model (models 1 and 2) (16)

(b) Assume that a particular trader uses a derivative pricing formula that involves the process $X_t = V_t^6$. Determine the SDE for X_t in terms of the parameters λ, β and μ and also an initial condition for X_t. (14)

3. Consider the Wilkie model. The real yield can be modelled by the equation

$$\ln(R(t)) = \ln(RMU) + RA[\ln(R(t-1)) - \ln(RMU)] + RBC.CE(t) + RE(t),$$

where $CE(t) = CSD.CZ(t)$, $RE(t) = RSD.RZ(t)$ and $CZ(t)$, $RZ(t)$ are series of i.i.d. standard normal random variables.

(a) Interpret each term in the equation and the global structure of the equation. (12)

(b) Explain why the variable modelled for the real yield is $\ln(R(t))$ and not $R(t)$; and list what are the parameters in the equation that must be estimated from data. (10)

4. Consider a 3-period recombining binomial model for the non-dividend paying share with price process S_t such that the price at time $t+1$ is either $S_t u$ or $S_t d$ with $d = \frac{1}{u}$. Assume that $u = 1.08$ and that the current price of the share is 10€.

(a) Discuss if the model is free of arbitrage (if it is not, prove that there is an arbitrage strategy) in the following cases:

i. if the risk-free interest rate is 0.02% per period (continuously compounded).

ii. if the risk-free interest rate is 10% per period (continuously compounded). (12)

(b) Assuming that the risk-free interest rate is 4% per year, construct the binomial tree and calculate the price of a derivative composed of a sum of an European put option and an European call option. The put option has strike $K_p = 8.5$ and the call option has a strike $K_c = 12$. The time to maturity of both options is 3 years. (18)

(c) Compare, from the computational effort point of view, a recombining binomial model of the type considered before with n periods (n large) with the general (non-recombining) binomial model with the same n periods. Moreover, explain what is the relationship between the form of the volatility and the recombining model and how you can calculate the factor u from the volatility parameter associated to the lognormal continuous model. (14)

5. Consider the Black-Scholes model and a stock currently priced at 10 Euros. The writer of 500000 European call options on this stock, with strike price 9.75 Euros and one year maturity, composed a hedging portfolio containing 400000 shares and a cash loan. Consider that the continuously compounded risk-free interest rate is 8% and that the share pays no dividends.

(a) From the Black-Scholes formula, derive the general expression for the Delta of a call option and calculate the Delta for this particular call option. (18)

(b) Calculate: (i) the implied volatility for the call option; (ii) the price of the call option; (18)

6. Consider the zero-coupon bond market.

(a) Discuss the limitations of one-factor interest rate models. (14)

(b) Assume that under the risk-neutral measure $\mathbb{Q}$, the dynamics for the instantaneous forward rate process is

$$df(t,T) = b(t,T)dt + v\,(t.T)\,dW_t,$$

and the dynamics for the zero coupon bond price is

$$dB\,(t.T) = B\,(t,T)\,[h(t,T)dt + S(t,T)dW_t]\,,$$

where

$$h(t,T) = r(t) - \int_t^T b(t,u)du + \left(\int_t^T v(t,u)du\right)^2,$$
$$S\,(t,T) = -\int_t^T v(t,u)du.$$

Prove that if the bond market is complete, then $h(t,T) = r(t)$ and

$$dB\,(t.T) = B\,(t,T)\left[r(t)dt - \left(\int_t^T v(t,u)du\right)dW_t\right].$$

(18)