Master in Actuarial Science

Models in Finance

12-01-2018

Time allowed: Two and a half hours (150 minutes)

Solutions

1. .

   (a) By Itô's lemma (or Itô's formula) applied to $f(t,x) = \exp(\sigma x + \frac{1}{2}(\mu - \sigma^2 - 2r)t)$ (it is a $C^{1,2}$ function):

$$\begin{aligned} d\widetilde{S}_t &= \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)\,(dB_t)^2 \\ &= \frac{1}{2}(\mu - \sigma^2 - 2r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t + \frac{1}{2}\sigma^2 \widetilde{S}_t dt \\ &= \left(\frac{\mu}{2} - r\right)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t. \end{aligned}$$

   where we have used $(dB_t)^2 = dt$. Therefore

$$d\widetilde{S}_t = \left(\frac{\mu}{2} - r\right)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t.$$

   (b) In general, the discounted price process $\widetilde{S}_t$ is not a martingale under the real world probability $\mathbb{P}$. Indeed, since in the SDE above, the drift coefficient $\left(\frac{\mu}{2} - r\right)\widetilde{S}_t$ is not zero, the process $\widetilde{S}_t$ is not a martingale.
   Under the equivalent martingale measure Q, the discounted price process $\widetilde{S}_t$ is a martingale, the drift coefficient is zero and the diffusion coefficient of the SDE remains the same, i.e.

$$d\widetilde{S}_t = \sigma \widetilde{S}_t d\overline{B}_t,$$

   where $\overline{B}_t$ is a standard Brownian motion under Q.

(c) Replacing the parameter values, we have

$$P\left(1.05<\frac{S_1}{S_0}<1.40\right)=P\left(ln(1.05)<\frac{1}{2}\left(0.16-0.15^2\right)+0.15B_1<ln(1.40)\right)$$

and

$$=P\left(0.0488<Z<0.3365\right)$$

where $Z=0.06875+0.15B_1\sim N\left(0.06875;0.0225\right)$.
Therefore: $P\left(1.05<\frac{S_1}{S_0}<1.40\right)=0.5158$.

2. .

(a) In the long term, $\mathbb{E}\left[V_\infty\right]=0.07+0.15\mathbb{E}\left[V_\infty\right]$ and therefore $\mathbb{E}\left[V_\infty\right]=0.0824$. Moreover, $Var\left[V_\infty\right]=(0.15)^2\,Var\left[V_\infty\right]+0.05^2$ and therefore $Var\left[V_\infty\right]=0.002558$. Since V_t is obtainded from $V_{t-1}$by summing one independent normal random variable, the process V_t is Gaussian and the long term distribution is $N\left[0.0824;0.002558\right]$. The model (2) corresponds to the Ornstein-Uhlenbeck process with mean reversion and the long term mean for this model is μ. The long term distribution is also a normal distribution. Therefore, we require $\mu=0.0824$. Moreover, for the model (2), the long term variance is given by $\frac{\beta^2}{2\lambda}$ and therefore we require that

$$Var\left[V_\infty\right]=\frac{\beta^2}{2\lambda}=0.002558.$$

(b) We just need to apply the Itô formula to the function $f(x)=x^6$in order to obtain

$$dX_t=6V_t^5dV_t+\frac{1}{2}30V_t^4\left(dV_t\right)^2=6V_t^5\left[-\lambda\left(V_t-\mu\right)dt+\beta dB_t\right]+\frac{1}{2}30V_t^4\beta^2dt$$

and therefore, we obtain

$$dX_t=\left(-6\lambda X_t+6\mu\lambda X_t^{\frac{5}{6}}+15\beta^2X_t^{\frac{4}{6}}\right)dt+6\beta X_t^{\frac{5}{6}}dB_t$$

and this is a SDE for the process X_t. The initial condition is $X_0=V_0^6=0.12^6=2.986*10^{-6}$.

3. .

(a) Global structure of the equation: this year's value = long run mean $(\ln(RMU))+RA$(last year's value − long run mean) + a stochastic shock to the system + another stochastic shock to the system.

The parameter RA is the autoregressive parameter (for the mean-reverting effect). The term $CE(t)$ is a stochastic shock from another process.

The term $CZ(t)$ is the random error term used to model conventional bond yields and $CZ(t)$ and $RZ(t)$ are not combined into a simple series of i.i.d. standard normal random variables because of the correlations that exist between conventional and index-linked bonds.

(b) The real yield of an index-linked bond $R(t)$ is positive, that is the reason why we model $\ln(R(t))$ and not $R(t)$ directly. The parameters to be estimated from data are: RMU, RA, RBC, CSD and RSD.

4. .

(a) $u = 1.08$ and $d = 1/u = 0.9259$

(i) The model is arbitrage free if and only if $d < e^r < u$. If $r = 0.0002$ then $e^r = 1.0002$. In this case, $d < e^r < u$ and the model is arbitrage free.

(ii) If $r = 0.1$ then $e^r = 1.1051$ and$u < e^r$. In this situation, the cash investment would outperform the share investment in all circumstances. An investor could (at time 0) sell the share and invest $S_0 = 10$ Euros in a cash account. At time 1 he could buy again the share and have a certain positive profit of $S_0 e^r - S_0 u = 10\exp(0.1) - 10 \times 1.08 = 0.2517 > 0$ or $S_0 e^r - S_0 d = 10\exp(0.1) - 10 \times 0.9259 = 1.7927 > 0$ (arbitrage opportunity).

(b) If $r = 4\%$, then the risk-neutral probability for an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.04} - 0.9259}{1.08 - 0.9259} = 0.7457.$$

Binomial tree values: 10; 10.8,9.259; 11.664, 10, 8.5729; 12.5971,10.8, 9.259, 7.9376

Payoff function of the derivative (call + put):

$$Payoff = \begin{cases} 8.5 - S_T & \text{if } S_T < 8.5 \\ 0 & \text{if } 8.5 \le S_T \le 12 \\ S_T - 12 & if S_T > 12 \end{cases}.$$

Payoff of the derivative: $C_3\left(u^3\right) = 12.5971 - 12 = 0.5971$, $C_3\left(u^2 d\right) = 0$, $C_3\left(ud^2\right) = 0$, $C_3\left(d^3\right) = 8.5 - 7.9376 = 0.5624$

Using the usual backward procedure with $r = 0.04$ and $q = 0.7457$:

At time 2: $C_2\left(u^2\right) = \exp(-r)\left[qC_3\left(u^3\right) + (1-q)C_3\left(u^2 d\right)\right] = 0.4278$,

$C_2\left(ud\right) = \exp(-r)\left[qC_3\left(udu\right) + (1-q)C_3\left(ud^2\right)\right] = 0$, $C_2\left(d^2\right) = \exp(-r)\left[qC_3\left(d^2u\right) + (1-q)C_3\left(d^3\right)\right] = 0.1374$
At time 1: $C_1(u) = \exp(-r)\left[qC_2\left(u^2\right) + (1-q)C_2\left(ud\right)\right] = 0.3065.$,
$C_1(d) = \exp(-r)\left[qC_2\left(du\right) + (1-q)C_2\left(d^2\right)\right] = 0.0336.$
The Final price (at time 0) is$C_0 = \exp(-r)\left[qC_1\left(u\right) + (1-q)C_1\left(d\right)\right] = 0.2278$

(c) The non-recombinig binomial model allows for different values of volatility when in different states (it allows different up and down factors for different states): $u_t\left(j\right)$ and $d_t\left(j\right)$ vary with t and j. Therefore, the number of states at time N is 2^N states: if N is large, it is a big number with exponential growth (for computational purposes), since computation times even for simple derivative securities are at best proportional to the number of states. For example, with 20 periods, at time $t = 20$ we have $2^{20} = 1048576$ states.

In the recombining binomial model, it is assumed that the volatility is the same at all states (the up and down factors are the same irrespective of wether they appear in the binomial tree) and all periods. At time N we have $N+1$ possible states (linear growth with N) instead of 2^N. For example, in a 20-period model, we have 21 states at time $t = 20$, instead of 1048576 states. Therefore, with this model the computing times are substantially reduced.

If we require the expected value and the variance of returns in one period of size δt of the Binomial model and of the continuous lognormal model to be equal, then we can deduce the following relationship $u = \exp\left(\sigma\sqrt{\delta t}\right)$, which allows the calculation of the u parameter from the volatility parameter.

5. Consider the Black-Scholes model and a stock currently priced at 10 Euros. The writer of 500000 European call options on this stock, with strike price 9.75 Euros and one year maturity, invested on a hedging portfolio containing 400000 shares and a cash loan. Consider that the continuously compounded risk-free interest rate is 8% and that the shares pay no dividends.

   (a) We can write the price of a call option as

$$c(S) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where $d_1 = \frac{ln(\frac{S}{K})+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$. Since Φ is the cumulative distribution function of the N(0,1) distribution, we have that

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}}exp(-\frac{x^2}{2}),$$

which is the p.d.f. of N(0,1). Therefore, we have that

$$\Delta = \frac{\partial c}{\partial S} = \Phi(d_1) + S\Phi'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-rT}\Phi'(d_2)\frac{\partial d_2}{\partial S}.$$

Calculation of the partial derivatives and using the p.d.f. of the N(0,1) distribution and noting that

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S},$$

one can deduce that

$$S\Phi'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-rT}\Phi'(d_2)\frac{\partial d_2}{\partial S}. = 0$$

and

$$\Delta = \frac{\partial c}{\partial S} = \Phi(d_1).$$

For the particular option considered, the Delta is such that: Number of Shares = $\Delta \times$number of call options, or

$$400000 = \Delta \times 500000,$$

and therefore

$$\Delta = 0.8.$$

(b) (i) the implied volatility for the call option: Since $\Delta = 0.8 = \Phi(d_1)$, . inverting the cumulative distribution function, we obtain $d_1 = 0.8416$ and

$$d_1 = \frac{ln\left(\frac{10}{9.75}\right) + \left(0.08 + \frac{1}{2}\sigma^2\right)}{\sigma} = 0.8416$$

and this equation is a quadratic equation:

$$\frac{1}{2}\sigma^2 - 08416\sigma + 0.1053 = 0.$$

There is only one solution that is less than 1: it is

$$\sigma = 0.1362,$$

so this is the value for the implied volatility. (ii) In order to calculate the option price, we use the formula

$$c(S) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

with the parameters given before, the implied volatility calculated previously and using the value

$$d_2 = d_1 - \sigma\sqrt{T} = 0.8416 - 0.1362 = 0.7054.$$

We obtain

$$c(10) = 10 \times 0.8416 - 9.75 \times e^{-0.08}\Phi\,(0.7054) = 1.1624.$$

6. .

(a) (1) if we look at historical interest rate data we can see that changes in the prices of bonds with different terms to maturity are not perfectly correlated as one would expect to see if a one-factor model was correct. Sometimes we even see, for example, that short-dated bonds fall in price while long-dated bonds go up.

(2) If we look at the long run of historical data we find that there have been sustained periods of both high and low interest rates with periods of both high and low volatility. Again these are features which are difficult to capture without introducing more random factors into a model. This issue is especially important for two types of problem in insurance: the pricing and hedging of long-dated insurance contracts with interest-rate guarantees; and asset-liability modelling and long-term risk-management.

(3) we need more complex models to deal effectively with derivative contracts which are more complex than, say, standard European call options. For example, any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated.

(b) If the bond market is complete then the discounted zero-coupon bond price $\widetilde{B}(t,T) = \exp\left(-\int_0^t r(s)\,ds\right) B(t,T)$ is a martingale with respect to the risk-neutral probability measure $\mathbb{Q}$. By the Itô formula applied to $f(t,x) = \exp\left(-\int_0^t r(s)\,ds\right) x$, and by the fundamental theorem of integral calculus, we have that

$$\begin{aligned} d\widetilde{B}(t,T) &= -r(t)\exp\left(-\int_0^t r(s)\,ds\right) B(t,T)\,dt + \exp\left(-\int_0^t r(s)\,ds\right) dB(t,T) \\ &= -r(t)\widetilde{B}(t,T)\,dt + \widetilde{B}(t,T)\left[h(t,T)dt + S(t,T)dW_t\right] \\ &= \widetilde{B}(t,T)\left[(h(t,T)-r(t))\,dt + S(t,T)dW_t\right]. \end{aligned}$$

In order to be a martingale, the drift coefficient must be zero, that is, $h(t,T) - r(t) = 0$. Therefore

$$\int_t^T b(t,u)du = \left(\int_t^T v(t,u)du\right)^2.$$

and replacing these expressions in the dynamics of the zero-coupon bond prices, we obtain:

$$dB(t.T) = B(t,T)\left[r(t)dt - \left(\int_t^T v(t,u)du\right) dW_t\right].$$