LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Master in Actuarial Science
Rate Making and Experience Rating

Exam 1, 05/01/2018, 9:00
Time allowed: 2:30

Instructions:

1. This paper contains 5 groups of questions and comprises 3 pages including the title page;
2. Enter all requested details on the cover sheet;
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so;
4. Number the pages of the paper where you are going to write your answers;
5. Attempt all questions;
6. Begin your answer to each of the 5 question groups on a new page;
7. Marks are shown in brackets. Total marks: 200;
8. Show calculations where appropriate;
9. An approved calculator may be used. No mobile phones or other communication devices are permitted;
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. The following table shows data generated by a certain automobile portfolio in some year corresponding to a *stable year* of exposure. The portfolio is supposedly homogeneous **[80]**

No. of Claims	0	1	2	3	4	Total
No. of policies	408,348	31,993	2,010	133	6	442,490

Let N be the number of claims per year for a given risk in the portfolio and suppose that $N \frown \text{Poisson}(\lambda)$. The parameter λ is unknown. Consider that the usual hypothesis in credibility theory may be applicable to the risk portfolio under study, λ is the associated risk parameter, it does not depend on the sum insured, claim number and claim sizes are independent and the expected value of the claim size is proportional to the sum insured. Observations from past years for the risk, and portfolio, are available.

Bühlmann's credibility (pure) premium for a given risk X in a homogeneous portfolio, for a coming year of exposure, is given by formula

$$P_c = z\bar{X} + (1-z)\mu_X,$$

where $z = n/(n+\upsilon/a)$, $\mu = E[\mu(\theta)]$, $\upsilon = E[\upsilon(\theta)]$, $a = Var[\mu(\theta)]$, $\mu(\theta)$ and $\upsilon(\theta)$ are the risk mean and variance, respectively, n is the number of years in force of that risk, and $\bar{X}$ is its sample mean.

(a) Consider two risks taken at random from the portfolio, say N_1 and N_2. Calculate the covariance. Do you agree with a classical statement saying that (15)

...the risks in the portfolio are independent?

Comment briefly.

(b) Consider the data above. Would you consider to come from a Poisson distributed population? If not, make a suggestion. Do a quick calculation to support both answers. (10)

(c) Comment briefly:

If you consider that the Poisson parameter is a realization of a (positive and non observable) random variable that means that the porfolio is no longer homogeneous. (5)

From now onwards admit that the parameter λ is a realization of a random variable Λ, non observable, following a Gamma distribution with mean α/β and variance $E[\Lambda]/\beta$.

(d) Show that the posterior distribution is of the same family of the prior, i.e. $\Lambda|N_1,\ldots,N_n$ follows a Gamma distribution with parameters $\alpha_* = \alpha + N_*$ and $\beta_* = (\beta+n)$, mean α_*/β_* and $N_* = \sum_{j=1}^{n} N_j$. (15)

(e) Although the risk parameter Λ is not observable you can estimate Prior's parameters using collective data available. Comment briefly and calculate corresponding estimates $\hat{\alpha}$ and $\hat{\beta}$. (15)

In addition, consider in what follows that a given risk belonging to the portfolio has produced 2 claims in a row in last five years.

(f) Compute the *Empirical Bayes Premium* for the coming rating year for the given risk. (15)

(g) Calculate the *Empirical Credibility Premium* for the same year and compare it to the previous one. Comment appropriately. (5)

2. Consider a portfolio where a certain risk X can produce at most a claim in each year, of a fixed amount of 100 units, with probability θ. Furthermore, θ can be considered as an outcome of a random variable Θ following a Beta distibution with parameters $\alpha = 0.4$ and $\beta = 0.6$.

For the last six years the risk has produced an average (aggregate) claim amount of 50 units. **[25]**

(a) Write and explain briefly all the model assumptions that you need to assume in order to estimate the next year premium for the risk. (7.5)

(b) Calculate the Bayesian premium. Show and explain clearly all the steps taken. (17.5)

3. Suppose you have observed for a certain risk X for n years, X_j, $j = 1, 2, \ldots, n$, in a portfolio. Consider Bühlmann's H_1 and H_2. The Credibility (pure) Premium of the risk for year $n+1$ is defined as the linear estimnator $\widetilde{\mu_{n+1}} = a + b\,\bar{X}$, $\bar{X} = \sum_{j=1}^{n} X_j$, where a and b are such that: **[15]**

$$\min Q = \mathbb{E}\left\{\left[\mu_{n+1}(\theta) - \widetilde{\mu_{n+1}}\right]^2\right\}.$$

Let a^* and b^* be the solutions, $\mu(\theta) = \mathbb{E}[X|\theta]$, $\upsilon(\theta) = \mathbb{V}[X|\theta]$.

Knowing that $a^* = (1 - b^*)\mathbb{E}[\mu(\theta)]$, using appropriate computation find that,

$$b^* = \frac{\mathbb{V}[\mu(\theta)]}{\frac{1}{n}\mathbb{E}[\upsilon(\theta)] + \mathbb{V}[\mu(\theta)]}$$

4. For a given motor insurance portfolioto, a certain insurer uses a *bonus-malus* system (BMS) to rate each individual risk . **[50]**

   Also, consider a system that evolves according to what is shown in Table 1

Starting level	Level occupied if 0 claim no. reported is	1	2	≥ 3
5	4	5	5	5
4	3	5	5	5
3	2	5	5	5
2	1	4	5	5
1	0	3	5	5
0	0	2	4	5

Table 1: Transition rules

   (a) Define $t_{ij}(k)$, such that

$$t_{ij}(k) = \begin{cases} 1 & \text{if policy transfers from } i \text{ to } j\,, \\ 0 & \text{otherwise,} \end{cases}$$

   if k claims are reported. Write the Transition Rules Matrix $\mathbf{T}(k) = [t_{ij}(k)]$, $k = 0, 1, \ldots$

   (b) Suppose now that Number of Claims is $Poisson(0.1)$ distributed. Build the one-step transition probability matrix.

   (c) Most insurers do not use credibility to build *BMS*'s. State briefly your reasons.

   (d) Go back to the system referred in Table 1. Suppose that the only bonus level is "Level 0". What would you do if you could only reach the bonus level with two consecutive years with no claims?

5. A working party is re-modelling a tariff for a given existing motor insurance portfolio. A wide variety of risk factors affecting both the claim count and size are being tested to have influence on different premium levels. **[30]**

   (a) Give three to four examples for each type of *a priori* and *a posteriori* classification variables in ratemaking.

   (b) Insurers usually model relativities instead of actual premiums. Explain briefly what that means.

   (c) Suppose that when modelling the "expected claim size" (say, X) the group came out with the following final model:

$$\begin{aligned} \ln X \;=\; & -2.62863 + .34211x_{2,2} + .27076x_{2,3} + .20249x_{4,2} - 0.12602x_{5,1} + .17053x_{7,3} + .23968x_{7,4} \\ & +.21130x_{8,2} + .77360x_{9,2} + .20062x_{9,3} + .28878x_{11,1} - 0.22840x_{11,3} - 0.50593x_{12,4}\,, \end{aligned}$$

   where $x_{i,j}$ corresponds for risk factor i and level j, where $i = 1, \ldots, 12$.

   Write the base premium for a risk with an exposure corresponding to the following months: January, February and March of a common year.

   (d) Since the pure premium is given by the product between the expected claim counts and the claim size expectation, it's obvious that the study group should build the tariff based on a multiplicative model. Comment briefly.

Solutions:

1. (a)
$$Cov(N_1, N_2) = E[Cov(N_1, N_2|\Lambda)] + Cov(E[N_1|\Lambda], E[N_2|\Lambda]) = 0 + V[\Lambda] > 0\,.$$
Obviously... the risks are not independent.

(b) Sample mean and variance are 0.082343 and 0.086614, aproximately. Variance is bit more than 5% higher than the mean, probably not Poisson, may be Negative Binomial, a test should be done.

(c) Partly true. We should assume some heterogeneity inside the portfolio, by assuming some characteristic may vary among risks in the portfolio, when we assume possible different values for the risk paramenter.

(d) Compute, with $f_N(x|\lambda) = e^{-\lambda}\lambda^x/x!$,
$$\pi\left(\lambda|\underline{N} = \underline{x}\right) = \frac{f_{\underline{N}}(\underline{x}|\lambda)\pi(\theta)}{\int_0^\infty f_{\underline{N}}(\underline{x}|\lambda)\pi(\lambda)d\lambda} \to Gamma(\alpha_*, \beta_*)\,, \quad \underline{x} = (x_1, ..., x_n)\,.$$
It's an *exact credibility* situation.

(e) The parameter λ comes *attached* to the observable risk r.v., then observations of the risks must bring some information on the *hidden* aspects, say λ, so we use collective data.

Estimation: $\mu_N = \alpha/\beta$, $\sigma_N^2 = V[\mu(\theta)] + E[\upsilon(\theta)] = (\alpha/\beta)(1 + 1/\beta)$. $\hat{\mu_N} = \bar{N} = 0.0823431038$, $\hat{\sigma_N^2} = 0.086613814$, then $\hat{\alpha} \simeq 1.587648$; $\hat{\beta} \simeq 19.280891$.

(f) For $N_* = 2$ and $n = 5$,
$$E(\Lambda|N_* = 2) = \frac{\hat{\alpha} + N_*}{\hat{\beta} + n} \simeq \frac{1.587648 + 2}{19.280891 + 5} \simeq 0.147756\,.$$

(g) This is an *exact credibility* situation, than credibility premium and Bayesian premium match.

2.

(a) This a pure Bayesian model, and an *exact credibility* situation. Since the claim amounts are fixed, the annual aggregate claim amount is $100\,X$. Just model X:

i. H_1: Given θ, $X_1, X_2, \ldots X_{n+1} \overset{iid}{\frown} f(x|\theta)$. Explain...
Also, θ is an outcome of a r.v. $\Theta \frown \pi(\theta)$. Explain...

ii. H_2: Risks $(X_1, \Theta_1), \ldots$ are independent and $\Theta_i \overset{d}{=} \Theta$.

(b) $X|\theta \frown B(1, \theta)$, $\Theta \frown Beta(0.4, 0.6)$, $\mu(\theta) = \theta$. These are conjugate distributions so, the posterior is of the same family and $\pi(\theta|\bar{x}) = Beta(6\bar{x} + 0.4, 0.6 + 6 - 6\bar{x})$, where $\bar{x} = 50/100 = 0.5$. Then the Bayesian premium comes
$$100\,E[\mu(\theta)|\underline{x}] = 100\,E[\theta|\underline{x}] = 100\,\frac{3 + 0.4}{6 + 1} = \frac{340}{7}\,.$$

3. Find the gradient directly, or use the fact that
$$Q = \mathrm{V}[\mu(\theta) - b\,\overline{X}] + \left(\mathrm{E}\left[\mu(\theta) - b\overline{X}\right] - a\right)^2\,,$$
then, you just need to minimize
$$\begin{aligned}
\mathrm{V}[\mu(\theta) - b\,\overline{X}] &= \mathrm{E}[\mathrm{V}[\mu(\theta) - b\,\overline{X}|\theta]] + \mathrm{V}[\mathrm{E}[\mu(\theta) - b\,\overline{X}|\theta]] \\
&= \frac{b^2}{n}\mathrm{E}[\upsilon(\theta)] + (1 - b)^2\mathrm{V}[\mu(\theta)]. \\
\mathrm{V}[\overline{X}|\theta] &= \frac{1}{n}\mathrm{V}[X_{ij}|\theta]
\end{aligned}$$
Differentiating w.r.t. b and equating,
$$\frac{2\,b}{n}\mathrm{E}[\upsilon(\theta)] - 2(1 - b)\mathrm{V}[\mu(\theta)] = 0\,,$$
solution follows.

4. (a)

$$
\boldsymbol{T}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{T}(1) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\boldsymbol{T}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \boldsymbol{T}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ for all } k \geq 3.
$$

(b)

$$
\boldsymbol{P}(0.1) = \begin{pmatrix} 0.904837 & 0 & 0.090484 & 0 & 0.004524 & 0.000155 \\ 0.904837 & 0 & 0 & 0.090484 & 0 & 0.004679 \\ 0 & 0.904837 & 0 & 0 & 0.090484 & 0.004679 \\ 0 & 0 & 0.904837 & 0 & 0 & 0.095163 \\ 0 & 0 & 0 & 0.904837 & 0 & 0.095163 \\ 0 & 0 & 0 & 0 & 0.904837 & 0.095163 \end{pmatrix}
$$

.

(c) Despite being fair and financially balanced, credibility systems give harsh penalties, managers and regulators do not like them...

(d) Split classes. If level zero is the only bonus class, we could only split level 1 into levels 1.a) and 1.b), where you reach level 1.b) from level 1.a) if you have no claims where you would wait for a 2nd consecutive year with zero claims.

5.

(a) *A priori*: Age class, sex, type of car, use of car, territory, traffic... *A posteriori*: deductibles, credibility, bonus-malus...

(b) It is a relative number, a multiplying factor to be referred to a base premium, of unit 1.

(c)

$$
\exp\{-2.62863\}\frac{90}{365} \sim 0.0237295\,.
$$

(d) It has nothing to do with that... You may be able to use an additive model.