Financial Markets and Instruments

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Suggested Solutions

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GROUP I (45 points)

1. What are efficient portfolios? Explain how you would find out the efficient frontier in a market with a large amount of risky assets, but no risk-free asset. . . . . [15p]
Answer:
An efficient portfolio is a portfolio that, for a given level of risk, has the highest possible expected return. Or, alternatively, that for a given level of expected return, minimizes risk.

   If there is no risk-free asset, then we know – by the envelope's theorem – the investment opportunity set of risky portfolios is bounded by an hyperbola. Given the mean-variance theory inputs, i.e. given a vector of expected returns $\bar{R}$ and a variance-covariance matrix V, finding two efficient portfolios is enough to determine the entire efficient frontier as any point in the hyperbola can be seen as a combination of these two points – two funds theorem.

   To find two efficient portfolios it is enough to take two tangents to the investment opportunity set. So the finding of the efficient frontier reduces to the following steps: (1) choose a fictitious risk-free return, (2) find the combination (portfolio $T1$) of risky assets that maximizes the Sharpe ratio assuming that fictitious risk free return level, (3) choose another risk-free return and repeat step 2 to find a second tangent portfolio (portfolio $T2$); (4) find the hyperbola that describes all combinations of the two efficient portfolios $T1$ and $T2$.

2. Explain the connection between the utility function $U(W)$, its risk tolerance function $f(\sigma_p, \bar{R}_p)$ and the associated indifference curves. Motivate the shape of the indifference curves of risk averse, risk neutral and risk lover investors in the plane $\left(\sigma_p, \bar{R}_p\right)$. . . . . [15p]
Answer:
The so called risk tolerance function $f(\sigma_p, \bar{R}_p)$ is nothing but the expected value of the utility function, so $f(\sigma_p, \bar{R}_p) = \mathbb{E}\left[U(W)\right]$. For investors who verify the Von Newman-Morgensten axioms the choice of the optimal portfolio can be made by maximizing the risk tolerance function over the efficient frontier. Whenever $f(\sigma_p, \bar{R}_p)$ exists in closed-form one can draw it on the plan $\left(\sigma_p, \bar{R}_p\right)$ using the associated indifferent curves, i.e., curves in constant expected utility level $f(\sigma_p, \bar{R}_p) = K$ for $K \in \mathbb{R}$.

   Risk lovers appreciate risk so much they would be willing to accept a decrease in expected return provided as risk increases. These investors would have decreasing indifference curves in the risk/expected return plan. Risk neutral investors take decisions based upon expected return only as they are immune with respect to risk levels. Thus, their indifference curves as horizontal lines in the risk/expected return plan. Finally risk averse investors, which constitutes the market's majority would require higher expected return to incur in additional risk, so the have increasing indifference curves in the risk/expected return plan.

. . . Sketch here . . .

3 Choose <u>ONE</u> of the following statements and discuss whether they are true or false. . . . . . [15p]

I. *Most return generating models are based upon unrealistic assumptions, thus, there is no sound ground for applying them in practice.*

Comment: FALSE

It is true that return generating models such as constant correlation models, single factor models or multi-factor models, rely on assumptions that may, or may not, hold in practice, thus their use may lead to the introduction of *model risk* in mean-variance optimisation. That, however, does not mean they are useless as they also contribute to reduce another important risk associated with the application of mean-variance theory - *estimation risk.*

Indeed, without any model, the application of mean-variance theory requires estimation of all its inputs: all expected returns, $\bar{R}_i$, all volatilitites σ_i and all possible correlations $\rho_i j$. When we consider n assets the number of parameters to estimate are

$$n + n + \frac{n(n-1)}{2} : \quad \bar{R}_i, \quad \sigma_i, \quad \rho_i j, \quad \forall \quad i = 1, \cdots, n \quad , \quad j \neq i, \quad j = 1, \cdots, n.$$

Note the number of expected returns and volatilities grows linearly with the number of assets to be considered, but the number of correlations grows in a quadratic way The number of correlations to be estimated is huge and even small error on each estimate may lead to quite different conclusions in the end.

Using return generating models, the number of parameters to estimate tends to be much smaller and to grow either linearly with the number of assets – that is the case of constant correlation models and single factor models – or to be proportional to $n \times K$ for a relatively small K that stands for the number of factors in multi-factor models. The parameters associated with return generating models are not only less but also easier to estimate in a robust way, so using models help eliminating part of the estimation risk which may more the compensate the possible introduction of model risk.

II. *The possibility of obtaining abnormal returns based upon inside information is inconsistent with the efficient market hypothesis.*

Comment: FALSE

It only goes against the hypothesis of market's efficiency in its strongest form, but not in the semi-strong or weak efficient sense.

Strong efficiency would require prices to include all possible information, including private information and, thus, protecting investors goes against this form of efficiency. However semi-strong efficiency requires prices would reflect only all possible public information, while weak efficiency requires prices reflect only historical information.

Existence of laws against inside trading is, therefore, compatible with semi-strong or weak efficient markets.

GROUP II (35 points)

Let $U(W) = ae^{bW}$, where W is wealth. If the investor prefers more to less and is risk averse,
(i) What can you say about a and b? .... [10p]
(ii) Assuming those parameters conditions hold what can you conclude about absolute and relative risk aversion. Interpret. .... [10p]
(iii) Show that the second order Taylor approximation to the risk tolerance function associated with $U(\cdot)$, is *equivalent* to

$$f(\sigma, \bar{R}) = \bar{R} + \frac{bW_0}{2}(\bar{R}^2 + \sigma^2) \ .$$

where W_0 is the initial wealth to be invested. .... [15p]

Solution:

(i) I If the investor prefers more to less, we need $U'(W) > 0$ and if he is risk averse, we also need $U''(W) < 0$

$$U(W) = ae^{bW} \qquad U'(W) = abe^{bW} > 0 \qquad U''(W) = ab^2e^{bW} < 0 \qquad \Longrightarrow \qquad a < 0, b < 0.$$

(ii) Assuming we indeed have $a < 0, b < 0$

$$\begin{aligned} A(W) &= -\frac{U''(W)}{U'(W)} = -\frac{ab^2e^{bW}}{ab\alpha e^{bW}} = -b & \qquad A'(W) &= 0 \\ R(W) &= WA(W) = -bW & \qquad R'(W) &= -b > 0 \quad \text{for} \quad b < 0 \end{aligned}$$

From the above formulas we can conclude the investor has constant absolute risk aversion and $-b > 0$ is his coefficient of absolute risk aversion. This means that if the money available to invest would increase he would not invest any additional amount in risky assets, nor would he disinvest. Consistently with a constant absolute risk aversion, the investor has a increasing relative risk aversion, which means that for increasing amounts of money he decides to risk lower percentages of that money in risky assets.

(iii) The risk tolerance function (RTF) associated with $U(\cdot)$ is nothing but $\mathbb{E}(U(W))$. The goal is to write it taking as domain the usual mean-variance space $\big(\sigma, \bar{R}\big)$. We start by taking a second-order Taylor approximation of the $U(W)$ around the initial investment W_0 and use $W = W_0(1 + R)$.

$$\begin{aligned} U(W) &\approx U(W_0) + U'(W_0)(W - W_0) + \frac{1}{2}U''(W)(W - W_0)^2 \\ &\approx ae^{bW_0} + abe^{bW_0}W_0R + \frac{1}{2}ab^2e^{bW_0}W_0^2R^2 \end{aligned}$$

Any linear transformation of the above expression with a positive "'slope" coefficient, leads to an equivalent RTF. Using the linear transformation

$$V(W) = -ae^{bW_0} + U(W)\underbrace{\left(abe^{bW_0}W_0\right)^{-1}}_{>0}$$

we can then conclude

$$\begin{aligned} E(U(W)) \quad \Leftrightarrow \quad E(V(W)) &\approx \mathbb{E}\left[R + \frac{1}{2}bW_0R^2\right] \\ &\approx \bar{R} + \frac{bW_0}{2}(\bar{R}^2 + \sigma^2) \ = f(\sigma, \bar{R}) \ . \end{aligned}$$

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GROUP III

Problem 1 (80 points)

In a particular market most analysts believe the following two-factor APT equilibrium holds,

$$\bar{R}_i^e = 0.03 + 0.14 b_{1i} + 0.02 b_{2i} \ .$$

for two factors with volatilities of 30% and 10%, respectively.

1. Discuss the main assumptions of APT equilibrium models and interpret the parameters of the equilibrium relationship above. .... [7.5p]
   Solution:
   The main assumptions of APT equilibrium models are: (i) that we know the model generating the returns (in this case a two-factor model); (ii) that the law of one price holds so that in equilibrium there cannot exist arbitrage portfolios; (iii) there exist a risk-free asset to both deposit and borrow; (iv) investors only care about expected returns and risk measured by volatility.
   The equilibrium relationship let us all assets are in the hyperplane in the space $\left(\bar{R}, b_1, b_2\right)$ and since the expression holds also for the risk-free asset and the factors we have

$$\begin{aligned} R_f^e &= 0.03 + 0.14 \times 0 + 0.02 \times 0 = \ 3\% \\ \bar{F}_1 &= 0.03 + 0.14 \times 1 + 0.02 \times 0 = 17\% \\ \bar{F}_2 &= 0.03 + 0.14 \times 0 + 0.02 \times 1 = \ 5\% \end{aligned}$$

   where we use the fact that the risk-free asset does not vary with any of the factors and that the factors are independent of one another.

2. Consider that in this world there are two extremely well diversified portfolios, A and B, and we know $b_{1A} = 0, b_{2A} = 2.4$ and $b_{1B} = 0.8, b_{2B} = 0$.

   (a) Show that the mean-variance theory inputs, when we take portfolios A and B as our basic assets[1], are

$$\bar{R} = \begin{pmatrix} 7.8\% \\ 14.2\% \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} 0.0576 & 0 \\ 0 & 0.0576 \end{pmatrix} \ .$$

   .... [7.5p]
   Solution:
   Using the equilibrium relationship we get the expected returns

$$\bar{R}_A = 0.03 + 0.14 \times 0 + 0.02 \times 2.4 = 7.80\% \ , \bar{R}_B = 0.03 + 0.14 \times 0.8 + 0.02 \times 0 = 14.2\% \ .$$

   Given the information that A and B are extremely well diversified portfolios, we know their specific variances are to be considered as zero and the total variance can be obtained by $\sigma_i^2 = b_{1i}^2 \sigma_{F1}^2 + b_{i2}^2 \sigma_{F2}^2$ and the covariance by $\sigma_{ij} = b_{1i} b_{1j} \sigma_{F1}^2 + b_{2i} b_{2j} \sigma_{F2}^2$:

$$\begin{aligned} \sigma_A^2 &= 0^2 \times 0.30^2 + (2.4)^2 \times 0.1^2 = 0.0576 && \Rightarrow \quad \sigma_A = 24\% \\ \sigma_B^2 &= 0.8^2 \times 0.30^2 + 0^2 \times 0 = 0.0576 && \Rightarrow \quad \sigma_B = 24\% \\ \sigma_{AB} &= 0 \times 0.8 \times 0.30^2 + 2.4 \times 0 \times 0.1^2 = 0 \end{aligned}$$

[1] In case you did not answer Question 1 take the factors' expected returns to be $\bar{F}_1 = 17\%$ and $\bar{F}_2 = 5\%$.

(b) Show that, in this case, the combination of A and B with the lowest possible volatility is the homogeneous portfolio. Find its expected return and volatility. .... [10p]

Solution:

Let us denote by x the weight of portfolio A and $(1-x)$ the weight of portfolio B. The variance of any combination of A and B is given by

$$\sigma_p = x^2\sigma_A^2 + (1-x)^2\sigma_B^2 + 2x(1-x)\sigma_{AB} \ .$$

To find the portfolio that has $\min \sigma_p$ we derive the above expression and set it to zero, so our FOC is

$$2x\sigma_A^2 - 2(1-x)\sigma_B^2 + 2(1-2x)\sigma_{AB} = 0 \qquad \Leftrightarrow \qquad x = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} \ .$$

Using the concrete values of the exercise we get $x = \dfrac{0.0576 - 0}{0.0576 + 0.0576} = 0.5$, and we can conclude the minimum variance portfolio is indeed the homogeneous portfolio.

Using the weights we get the minimum variance portfolio expected return and volatility

$$\bar{R}_{MV} = \frac{1}{2} \times 7.8\% + \frac{1}{2}14.2\% = 11\%$$

$$\sigma_{MV}^2 = \left(\frac{1}{2}\right)^2 \times 0,057600 + \left(\frac{1}{2}\right)^2 \times 0,057600 = 0.0288 \quad \Rightarrow \quad \sigma_{MV} = 16.97\%$$

(c) Consider now all possible combinations of A and B and assume their returns are Gaussian.

(i) Determine the safest combination S, according to Roy's safety criteria when $R_L = 3\%$. Motivate your computations. .... [10p]

Solution:

The safest combination according to Roy and with $R_L = 3\%$ is the the one that solves $\min\limits_p \Pr(R_p < 3\%)$. For Guassian returns we have

$$\Pr(R_p < 3\%) = \Pr\left(\frac{R_p - \bar{R}_p}{\sigma_p} < \frac{3\% - \bar{R}_p}{\sigma_p}\right) = \Phi\left(\frac{3\% - \bar{R}_p}{\sigma_p}\right)$$

and thus

$$\min_p \Pr(R_p < 3\%) \quad \Leftrightarrow \quad \min_p \frac{3\% - \bar{R}_p}{\sigma_p} \quad \Leftrightarrow \quad \max_p \frac{\bar{R}_p - 3\%}{\sigma_p}$$

We know that from FOC of maximising such ratios we have

$$\left[\bar{R} - 3\%\right] = \begin{pmatrix} 4.8\% \\ 11.2\% \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} 0.05761 & 0 \\ 0 & 0.05761 \end{pmatrix}, \qquad \text{thus,}$$

$$Z = V^{-1}\left[\bar{R} - 3\%\right] = \begin{pmatrix} 17.3611 & 0 \\ 0 & 17.3611 \end{pmatrix} \begin{pmatrix} 4.8\% \\ 11.2\% \end{pmatrix} = \begin{pmatrix} 0.8333 \\ 1.9444 \end{pmatrix} \qquad \Rightarrow \qquad X \approx \begin{pmatrix} 30\% \\ 70\% \end{pmatrix}.$$

(ii) Determine the expected return and volatility of combination S. .... [5p]

Solution:

$$\bar{R}_S = \begin{pmatrix} 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 7.8\% \\ 14.2\% \end{pmatrix} = 12.28\% \qquad \text{and} \tag{1}$$

$$\sigma_S^2 = \begin{pmatrix} 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 17.3611 & 0 \\ 0 & 17.3611 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} = 0.03341 \quad \Rightarrow \quad \sigma_S = 18.28\% \tag{2}$$

(d) Suppose that besides investing in A and B, you can also lend or borrow at $R_f = 3\%$.

(i) Explain and sketch the shape of the investment opportunity set and the efficient frontier in $(\sigma, \bar{R})$ space. .... [10p]

Solution:

The risky portfolios A and B have the same risk but different expected returns and are independent of one another. Combinations of the risky portfolios A and B are described by an hyperbola passing trough A and B and with a minimum variance point at the homogeneous portfolio (exactly at the average expected return).

If we have an $R_f = 3\%$ that can be used to both lend and borrow, than combinations of the riskless asset with any hyperbola point can be represented by two straight lines, one passing in the riskless point $(0\%, 3\%)$ and that hyperbola point, and a second one also passing though the riskless point $(0\%, 3\%)$ but with symmetric slope.

The entire investment opportunity set is the open cone with two symmetric limiting lines departing from $(0\%, 3\%)$. The upper one is the efficient frontier, it goes trough t $(0\%, 3\%)$ and is tangent from above to the hyperbola.

... Sketch here ...

(ii) Show that, in this case, portfolio S from c(i) is also the only combination of A and B that remains efficient. Write down the equation for the efficient frontier. .... [7.5p]

Solution:

Recall from 3.(i) that our portfolio S is the one that $\max_p \dfrac{\bar{R}_p - 3\%}{\sigma_p}$, and for $R_f = 3\%$ it is therefore the combination that maximising the Sharpe Ratio. So S is our tangent portfolio and, thus the only combination of just risky assets that remain efficient after the introduction of the riskless asset.

The equation for the efficient frontier is:

$$\bar{R}_p = R_f + \frac{\bar{R}_T - R_f}{\sigma_T}\sigma_p \quad \Longrightarrow \quad \bar{R}_p = 0.03 + \frac{12.28\% - 3\%}{18.28\%}\sigma_p \ \Leftrightarrow \ \bar{R}_p = 0.03 + 0.0577\sigma_p$$

3. The indifference curves of Mr. MindChanger's risk tolerance function are given by

$$\bar{R}_p = \sigma_p^2 - 2\sigma_p + K \ ,$$

for some constant K. He is considering investing in A, B and the riskless asset (from Question 2) and asked for your advice.

(a) What can you conclude about Mr. MindChanger risk profile? Explain and sketch his indifference curves in space $(\sigma, \bar{R})$ .... [7.5p]

Solution:

From the indifference curves of Mr. MindChanger we get

$$\left(\frac{\partial \bar{R}_p}{\partial \sigma_p}\right)_{IC} = 2\sigma_p - 2 \quad \Longrightarrow \quad \begin{cases} < 0 & \text{for} \quad \sigma_p < 1 \\ = 0 & \text{for} \quad \sigma_p = 1 \\ > 0 & \text{for} \quad \sigma_p > 1 \end{cases} .$$

We can thus conclude that up to volatility levels of 100% this investor is risk lover. But for extremely high risk levels (above 100%) he becomes risk averse.

. . . Sketch here . . .

(b) Assuming he could invest without restrictions. What would you recommend? Quantify and interpret your answer. . . . . [10p]

Solution:

From Question 2.(d) we know the efficient frontier without restrictions is

$$\bar{R}_p = 0.03 + 0.0577\sigma_p \ .$$

Mr. MindChanger optimal investment happens at the point when the slopes of the indiference curves and the efficient frontier match.

$$\begin{aligned} \left(\frac{\partial \bar{R}_p}{\partial \sigma_p}\right)_{IC} &= \left(\frac{\partial \bar{R}_p}{\partial \sigma_p}\right)_{EF} \\ 2\sigma_p^* - 2 &= 0.0577 \\ \sigma_p^* &= 125.40\% \end{aligned}$$

As expected the optimal risk level happens in the area of risk aversion. To have a risk level of 125.40% he must invest $x = \frac{125.40\%}{18.28\%} = 686\%$ in the tangent portfolio and, therefore take a loan of 586%. The final investment composition is

$$x_f = -586\% \ , \qquad x_A = 686\% \times 30\% = 205,8\% \ , \qquad x_B = 686\% \times 70\% = 480.2\%$$

(c) What if he cannot borrow nor shortsell? Explain. . . . . [5p]

Solution:

If he cannot borrow nor shortsell then the efficient frontier is the straightline up to T and then the hyperbola, provided it remain in the no-shortselling area. In the upper part of the hyperbola we invest more in B than in A – recall the minimum variance portfolio is the homogeneous portfolio and B has the same risk but higher expected return than A.

All feasible investments – without shorlltselling or borrowing – have a risk σ_p lower than the risk of B, since the hyperbola stops at B, i.e. $\sigma_p \leq 24\%$.

Since our investor is risk lovers for $\sigma_p < 100\%$, the best for him is to invest everything in B and get the maximum possible risk $\sigma_B = 24\%$:

$$x_f = -0\% \ , \qquad x_A = 0\% \ , \qquad x_B = 100\% \ .$$

Problem 2 (40 points)

Consider the following returns associated with two alternative investments X and Y.

Scenario	Prob	X Return	Y Return
A	1/3	0%	5%
B	1/3	5%	5%
C	1/3	10%	5%

1. Show that not all investors who prefer more to less choose Y over X, but all risk averse investors do. Explain. .... [15p]
   [Hint: Think in terms of stochastic dominances.]
   Solution:
   To show that not all investors who prefer more to less choose Y over X we need to show there are no first order dominance. To show risk averse investors prefer Y to X, we need to shw Y stochastically dominates X in second order terms. The table below show exactly that:

Returns	Pr X	Pr Y	Dist X	Dist Y	Sum Dist X	Sum Dist Y
0%	1/3	0	1/3	0	1/3	0
5%	1/3	1	2/3	1	1	1
10%	1/3	0	1	1	2	2
			no first order dominance		second order dominance: $Y \succ X$	

2. Take an initial investment of $W_0 = 5\ 000$ euros and consider Mr. Caution risk tolerance function

$$f(\sigma, \bar{R}_p) = 1000 \times \left(\bar{R}_p - 2\sigma_p^2\right) \ .$$

   (a) Check that indeed Ms.Caution prefers $Y \succ X$. Explain. .... [10p]
   Solution:
   From the shape of her risk tolerance function we know she is risk averse as it is increasing in $\bar{R}_p$ and decreasing σ_p, so she likes expected return but dislikes risk and by the previous result we expect him to prefer Y to X.
   To check that this is so, we start by computing the expected return and volatility associated with each investment: $\bar{R}_X = 5\%$, $\sigma_X = 4.08\%$, $\bar{R}_Y = 5\%$, $\sigma_Y = 0\%$ Note that

$$\mathbb{E}(U(X)) = f(4.08\%, 5\%) = 1000\left((5\%) - 2(4.08\%)^2\right) = 46.67$$
$$\mathbb{E}(U(Y)) = f(0\%, 5\%) = 1000\left((5\%) - 2(0\%)^2\right) = 50$$

   and we conclude that indeed $Y \succ X$.

   (b) Find out the certainty equivalent and risk premia associated with investment X. ...[7.5p]
   Solution:
   We need to find the certain return that makes him indifferent between X and that certain return. So we set

$$f(0\%, R_C) = \mathbb{E}(U(X)) \quad \Leftrightarrow \quad 1000 R_C = 46.67 \quad \Leftrightarrow \quad R_C = 4.67\%.$$

   When applied to $W_0 = 5\ 000$ euros we a certainty equivalent of $C = 5000 \times (1 + 4.67\%) = 5\ 233.33$ euros. The expected W if we invest in X is instead $\mathbb{E}(W) = 5000 \times (1 + \bar{R}_X) = 5000 \times (1 + 5\%) = 5\ 250$ euros. So we conclude the risk premium associated with and investment of 5 000 euros in X is $\pi = 5\ 250 - 5\ 233.33 = 16.67$ euros.

3. Do you think Ms. Caution should one also consider combinations of X and Y (instead of just seeing them as alternatives) ? If so what is his optimal combination ? .... [7.5p]

Solution:

Since X is a risky investment and Y is riskfree we know all combinations of X and Y are on a line in $(\sigma_p, \bar{R}_p)$ given by the equation

$$\bar{R}_p = R_Y + \frac{\bar{R}_X - R_Y}{\sigma_X}\sigma_p \qquad \Leftrightarrow \qquad \bar{R}_p = 0.05$$

note this happens because both have the same expected return, so all combinations have a 5% expected return as well. For a fixed level of return he should minimize risk and since Y has zero risk the optimal for her is not to do any combination but to invest 100% in Y.