Mathematical Finance

Lisbon, November 20, 2017

All notation must be clear. Arguments should be complete.

1. Consider a standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

   and a simple T-claim X of the form

$$X = \Phi(S_T)$$

   Derive the Black-Scholes PDE for the pricing function $F(t, s)$ including the boundary condition. Motivate the arbitrage argument carefully.

2. Consider again the Black-Scholes model in problem 1 above, as well as a T-claim Z of the form

$$Z = \Phi(S_T)$$

   Denote the pricing function of the Z-claim by $F(t, s)$.

   Now consider a T-claim X of the form

$$X = S_T \cdot \Phi(S_T)$$

   with Φ as above, and denote the pricing function of the X-claim by $G(t, s)$.

   Your task is to derive a formula which allows you to compute the function G in terms of the function F (like we did for an asset with a continuous dividend yield).
   Hint: A change of numeraire could be useful. You are allowed to use results from the theory of numeraire changes without proof.

3. Consider a model for two countries. We then have a domestic market (Portugal) and a foreign market (Japan). The domestic and foreign interest rates, r_d and r_f, are assumed to be given real numbers. Consequently, the domestic and foreign savings accounts satisfy

$$B_t^d = e^{r_d t}, \qquad B_t^f = e^{r_f t},$$

where B^d and B^f are denominated in units of domestic and foreign currency, respectively. The exchange rate process X, which is used to convert foreign payoffs into domestic currency (the "Euro/Yen"-rate), is modelled by the following stochastic differential equation under the objective measure P

$$dX_t = \mu X_t dt + \sigma X_t d\bar{W}_t,$$

where μ and σ are assumed to be constants and $\bar{W}$ is a P-Wiener process.

A *domestic martingale measure*, Q^d, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of domestic currency and discounted using the domestic risk-free rate, martingales. We assume that if you buy the foreign currency this is immediately invested in a foreign bank account. All markets are assumed to be frictionless.

(a) Derive the Q^d-dynamics of X.

(b) Now take the viewpoint of a foreign-based investor, that is an investor who consistently denominates her profits and losses in units of foreign currency. A *foreign martingale measure*, Q^f, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of foreign currency and discounted using the foreign risk-free rate, martingales.
Find the Girsanov transformation between Q^d and Q^f.

(c) The domestic (foreign) market is said to be *risk neutral* if the domestic (foreign) martingale measure is equal to the objective measure P. Under which conditions are both markets risk neutral?

4. Consider a standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

and let X_t denote the value process of a self financing portfolio without consumption. Solve the problem of maximizing log utility of terminal wealth, i.e. you want to maximize

$$E^P \left[\ln \left(X_T \right) \right]$$