Probability Theory and Stochastic Processes

Solutions

Jan 15, 2016

1. b) 0

2. 2

3. e^{-1}

4.

$$E(X|Y)(\omega) = \begin{cases} \omega + \frac{1}{4}, & \omega < \frac{1}{2} \\ \omega - \frac{1}{4}, & \omega \geq \frac{1}{2} \end{cases}$$

5. a) Recurrent non-null, period 1

b) Unique stationary distribution $(\frac{1}{a}, \ldots, \frac{1}{a})$, mean recurrence time a for all states.

6. Yes

Feb 1, 2016

1. a) 0

b) $\{\emptyset, \Omega, X^{-1}(\{a\}), X^{-1}(\{b\})\}$

c) We don't know

2. 0

3. a) states 1, 2: transient period=2; states 3,4: recurrent positive period=2

b) $(0, 0, 1/2, 1/2)$, $(+\infty, +\infty, 2, 2)$

4. a) not a martingale

b) $-\infty$

Jan 18, 2017

1. a)

$$F(x) = \begin{cases} 1, & x \geq \sqrt{2} \\ 0, & x < \sqrt{2} \end{cases}$$

$\phi(t) = e^{it\sqrt{2}}$. The distribution is the Dirac measure on $\mathbb{R}$ at $\sqrt{2}$.

b) Any that is equal to X a.e. Ex: $Y(x) = \sqrt{2}$.

2. Dirac distribution at 0.

3.

a) 1,2,3 transient; 4 positive recurrent

b) 1

c) $\pi = (0, 0, 0, 1)$, $\mu = (+\infty, +\infty, +\infty, 1)$

4.

a) not a martingale

b) $-\infty$

Feb 3, 2017

1. a)

$$F(x) = \begin{cases} 1, & x \geq 2 \\ x/2, & 0 \leq x < 2 \\ 0, & x < 0 \end{cases}$$

$\phi(t) = (e^{2it} - 1)/(2it)$, $t \neq 0$, $\phi(0) = 1$. The distribution is the Lebesgue measure on $[0, 2]$.

b) Any that is equal to X a.e.

2. 1/2

3. a) 1 transient, 2,3,4,5 positive recurrent

b) Per(1)=1, Per(2)=Per(3)=Per(4)=Per(5)=2

c) (0,1/6,1/6,1/3,1/3), $(+\infty, 6, 6, 3, 3)$

4. a) Yes

b) 1, 4/7, 4/7

Jan 17, 2018

1. a)

$$F(x) = \begin{cases} 1, & x \geq 0 \\ x + 1, & -1 \leq x < 0 \\ 0, & x < -1 \end{cases}$$

$\phi(t) = (1 - e^{-it})/(it)$, $t \neq 0$, $\phi(0) = 1$. The distribution is the Lebesgue measure on $[-1, 0]$.

b) Any that is equal to X a.e.

2. b) 3/4

3. a) 1 positive recurrent, 2,3,4 transient. Per(1)=1=Per(4), there are no periods for 2 and 3.

b) (1,0,0,0), $(1, +\infty, +\infty, +\infty)$

c) 1

4. $E(X_1)$

Feb 2, 2018

1. a) No. E.g. $\Omega \notin \mathcal{A}$.

b) $\sigma(\mathcal{A}) = \{A \subset \Omega \colon A \text{ is countable or } A^c \text{ is countable}\}$

3. a) 2,3 transient, 1,4 positive recurrent, Per(1)=Per(2)=Per(3)=Per(4)=1

b) Stationary distributions: $(a, 0, 0, 1-a)$ for any $0 \leq a \leq 1$; mean recurrence times: $(1, +\infty, +\infty, 1)$.

c) 1

4. $E(X_1)$