Formulário auxiliar Probabilidades MAEG 2017

PROBABILIDADE

- $P(A \cap B \cap C) = P(A)P(B \mid A)P(C \mid A \cap B)$
- Sendo $\{A_1, A_2, ..\}$ uma partição do espaço dos resultados com $P(A_j) > 0$, $j = 1,2,...$,

$$P(B) = \sum_j P(A_j)P(B \mid A_j) \quad ; \quad P(A_j \mid B) = \frac{P(A_j)P(B \mid A_j)}{\sum_i P(A_i)P(B \mid A_i)}.$$

VALOR ESPERADO, MOMENTOS E PARÂMETROS

	Discretas	Contínuas
$E[\psi(X)] =$	$\sum_x \psi(x) f_X(x)$	$\int_{-\infty}^{+\infty} \psi(x) f_X(x)\, dx$
$E[\psi(X,Y)] =$	$\sum_x \sum_y \psi(x,y) f_{X,Y}(x,y)$	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi(x,y) f_{X,Y}(x,y)\, dx\, dy$
$E[\psi(X,Y) \mid X = x] =$	$\sum_y \psi(x,y) f_{Y\mid X=x}(y)$	$\int_{-\infty}^{+\infty} \psi(x,y) f_{Y\mid X=x}(y) dy$

Momentos de ordem k	$\mu'_k = E(X^k)$	$\mu_k = E[(X-\mu)^k]$
Momentos de ordem *r+s*	$\mu'_{rs} = E(X^r Y^s)$	$\mu_{rs} = E\{(X-\mu_X)^r (Y-\mu_Y)^s\}$

$\mathrm{Var}(X) = E(X-\mu)^2 = E(X^2) - \mu^2$;

$\mathrm{Cov}(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\} = E(XY) - E(X)E(Y)$; $\quad \rho_{X,Y} = \dfrac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$

$\mathrm{E}(aX+bY) = aE(X) + bE(Y)$ e $\mathrm{Var}(aX+bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y)$ com *a*, *b* constantes

$E(Y) = E_X[E(Y \mid X)]$; $\quad \mathrm{Var}(Y) = \mathrm{Var}_X[E(Y \mid X)] + E_X[\mathrm{Var}(Y \mid X)]$

Coeficiente de assimetria: $\gamma_1 = \dfrac{\mu_3}{\sigma^3}$; *Kurtosis*: $\beta_2 = \dfrac{\mu_4}{\sigma^4}$

Quantil (caso contínuo): $\xi_\alpha : \int_{-\infty}^{\xi_\alpha} f(x)dx = \alpha \Leftrightarrow F(\xi_\alpha) = \alpha$

Função geradora de momentos: $M_X(s) = E(e^{sX})$; $E(X^r) = M_X^{(r)}(0)$

Desigualdades

- Desigualdade de Markov – Seja $\psi(X) \geq 0$ função (mensurável) da v.a. X; Se existir $E(\psi(X))$, vem, para qualquer número real $c > 0$, $P(\psi(X) \geq c) \leq \dfrac{E(\psi(X))}{c}$.

- Desigualdade de Chebychev – Se X é uma v.a. com média μ e variância σ^2 vem para qualquer $t > 0$, $P(|X-\mu| \geq t\sigma) \leq 1/t^2$

DISTRIBUIÇÕES TEÓRICAS

- **UNIFORME (DISCRETA)**

Caso $f(x) = \frac{1}{n}, \quad x = 1,2,...,n$; $\quad E(X) = \frac{n+1}{2}$; $\text{Var}(X) = \frac{n^2-1}{12}$; $M(s) = E(e^{sX}) = \begin{cases} \frac{e^s(1-e^{sn})}{n(1-e^s)} & s \neq 0, \\ 1 & s = 0. \end{cases}$

Caso $f(x) = \frac{1}{m+1}, \quad x = 0,1,2,...,m$; $E(X) = \frac{m}{2}$; $\text{Var}(X) = \frac{m(m+2)}{12}$; $M(s) = \begin{cases} \frac{1-e^{s(m+1)}}{(1+m)(1-e^s)} & s \neq 0, \\ 1 & s = 0. \end{cases}$

- **BINOMIAL** $X \sim B(n;\theta)$, $(0 < \theta < 1)$

$f(x \mid \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$, $x = 0, 1, 2, ..., n$

$E(X) = n\theta$; $\text{Var}(X) = n\theta(1-\theta)$;

Propriedades:

- $X \sim B(n;\theta) \Leftrightarrow (n - X) \sim B(n; 1-\theta)$
- $X_1 \sim B(n_1;\theta,)$, $X_2 \sim B(n_2;\theta)$, X_1 e X_2 independentes $\Rightarrow X_1 + X_2 \sim B(n_1 + n_2, \theta)$

- **BERNOULLI** $X \sim B(1;\theta)$

- **GEOMÉTRICA** $X \sim BN(1;\theta)$, $(0 < \theta < 1)$

$f(x \mid \theta) = (1-\theta)^{x-1}\theta$, $x = 1, 2, \ldots$

$E(X) = \frac{1}{\theta}$; $\text{var}(X) = \frac{1-\theta}{\theta^2}$; $\gamma_1 = (2-\theta)/\sqrt{1-\theta}$; $M(s) = \frac{\theta e^s}{1-(1-\theta)e^s}$

Propriedades:

- $P(X > s + t \mid X > t) = P(X > s)$

- **BINOMIAL NEGATIVA** $X \sim BN(1;\theta)$, $(0 < \theta < 1)$

$f(x \mid r, \theta) = \binom{x-1}{r-1} \theta^r (1-\theta)^{x-r}$, $x = r, r+1, r+2, \ldots$

$E(X) = \frac{r}{\theta}$; $\text{var(X)} = \frac{r(1-\theta)}{\theta^2}$; $\gamma_1 = \frac{2-\theta}{\sqrt{r(1-\theta)}}$; $M(s) = \left(\frac{\theta e^s}{1-(1-\theta)e^s} \right)^r$

Propriedades:

- $X_i \sim BN(k_i;\theta), i = 1,2,\cdots,m, X_i, X_j$ indep. $\Rightarrow Y = \sum_{i=1}^m X_i \sim BN\left(\sum_{i=1}^m k_i;\theta\right)$

- **HIPERGEOMÉTRICA** $X \sim H(M, N, p)$, $(0 < p < 1)$

$f(x \mid M, N, p) = \frac{\binom{Mp}{x}\binom{M(1-p)}{N-x}}{\binom{M}{N}}$, $x = 0, 1, \cdots, N$

$E(X) = Np$; $\text{var(X)} = Np(1-p)\frac{M-N}{M-1}$;

- **MULTINOMIAL** $X \sim BN(1;\theta)$, $(0 < \theta < 1)$

$f_{\mathbf{X}}(x_1, x_2, \ldots, x_k) = \frac{n!}{x_1!x_2!\ldots x_k!(n-x_1-x_2-\cdots-x_k)!} \theta_1^{x_1}\theta_2^{x_2}\ldots\theta_k^{x_k}(1-\theta_1-\theta_2-\cdots-\theta_k)^{n-x_1-x_2-\cdots-x_k}$,

$E(X_i) = n\theta_i$; $\text{var}(X_i) = n\theta_i(1-\theta_i)$; $\text{cov}(X_i, X_j) = -n\theta_i\theta_j$;

- **POISSON** $X \sim \text{Po}(\lambda)$, $(\lambda > 0)$

$f(x \mid \lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0,1,2,...$; $E(X) = \lambda$; $\text{Var}(X) = \lambda$; $M(s) = \exp\{\lambda(e^s - 1)\}$

Propriedades:

- $X_1 \sim \text{Po}(\lambda_1)$, $X_2 \sim \text{Po}(\lambda_2)$, X_1 e X_2 independentes $\Rightarrow X_1 + X_2 \sim \text{Po}(\lambda_1 + \lambda_2)$
- X_1 e X_2 independentes , $X_1 \sim \text{Po}(\lambda_1)$, $X_1 + X_2 \sim \text{Po}(\lambda_1 + \lambda_2) \Rightarrow X_2 \sim \text{Po}(\lambda_2)$
- Se $X \sim B(n;\theta)$, com n grande θ pequeno então $X \overset{a}{\sim} \text{Po}(n\theta)$

- **UNIFORME (CONTÍNUA)** $X \sim U(\alpha, \beta)$, $(\alpha < \beta)$

$f(x \mid \alpha, \beta) = \dfrac{1}{\beta - \alpha}$ $\alpha < x < \beta$; $E(X) = \dfrac{\alpha + \beta}{2}$; $\text{Var}(X) = \dfrac{(\beta - \alpha)^2}{12}$; $M(s) = \begin{cases} \dfrac{e^{s\beta} - e^{s\alpha}}{s(\beta - \alpha)} & s \neq 0 \\ 1 & s = 0 \end{cases}$

- **NORMAL** $X \sim N(\mu, \sigma^2)$**,** $(-\infty < \mu < +\infty, \quad 0 < \sigma < +\infty)$

$f(x \mid \mu, \sigma^2) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\dfrac{1}{2\sigma^2}(x - \mu)^2 \right\}$, $-\infty < x < +\infty$

$E(X) = \mu$; $\text{Var}(X) = \sigma^2$; $M(s) = \exp\left\{ \mu s + \dfrac{\sigma^2 s^2}{2} \right\}$

Propriedades:

- Normal estandardizada $Z = \dfrac{X - \mu}{\sigma} \sim N(0,1)$; $\phi(z) = \phi(-z)$ e $\Phi(z) = 1 - \Phi(-z)$
- $X_i \sim N(\mu_i, \sigma_i^2)$ (i = 1, 2, ..., k) independentes $\Rightarrow \sum_{i=1}^{k} \alpha_i X_i \sim N(\mu_Y, \sigma_Y^2)$ com $\mu_Y = \sum_{i=1}^{k} \alpha_i \mu_i$ e $\sigma_Y^2 = \sum_{i=1}^{k} \alpha_i^2 \sigma_i^2$
- X_1 e X_2 independentes, $X_1 + X_2 \sim n(\mu, \sigma^2) \Rightarrow X_1 \sim$ normal e $X_2 \sim$ normal

- **LOGNORMAL** $X \sim Lognormal(\mu, \sigma^2)$**,** $(-\infty < \mu < +\infty, \quad 0 < \sigma < +\infty)$

$f(x \mid \mu, \sigma^2) = \dfrac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(\ln x - \mu)^2}{2\sigma^2} \right)$, $x > 0$

$E(X) = \exp(\mu + \sigma^2 / 2)$; $\text{Var}(X) = \exp\left(2\mu + \sigma^2\right)\left(e^{\sigma^2} - 1\right)$;

Propriedades:

- $X \sim Lognormal(\mu, \sigma^2) \Leftrightarrow \ln X \sim n(\mu, \sigma^2)$

- **EXPONENCIAL** $X \sim \text{Ex}(\lambda)$**,** $(\lambda > 0)$; $X \sim \text{Ex}(\lambda) \Leftrightarrow X \sim \text{G}(1, \lambda)$

$f(x \mid \lambda) = \lambda e^{-\lambda x}$, $x > 0$ $E(X) = \dfrac{1}{\lambda}$; $\text{Var}(X) = \dfrac{1}{\lambda^2}$; $M(s) = \dfrac{\lambda}{\lambda - s}$

Propriedades:

- $X_i \sim \text{Ex}(\lambda)$ (i = 1, 2, ..., k) independentes $\Rightarrow$ $\sum_{i=1}^{k} X_i \sim G(k, \lambda)$ e $\min_i X_i \sim \text{Ex}(k\lambda)$

- **GAMA** $X \sim G(\alpha, \lambda)$**,** $(\lambda > 0, \alpha > 0)$

$f(x \mid \alpha, \lambda) = \dfrac{\lambda^\alpha e^{-\lambda x} x^{\alpha - 1}}{\Gamma(\alpha)}$, $x > 0$; $E(X) = \dfrac{\alpha}{\lambda}$; $\text{Var}(X) = \dfrac{\alpha}{\lambda^2}$; $M(s) = \left(\dfrac{\lambda}{\lambda - s} \right)^\alpha$

Propriedades:

- $X_i \sim G(\alpha_i; \lambda)$, (i = 1, 2, ..., k) independentes $\Rightarrow \sum_{i=1}^{k} X_i \sim \text{G}\left(\sum_{i=1}^{k} \alpha_i ; \lambda \right)$
- $X \sim G(\alpha, \lambda)$ então $Y = cX \sim G(\alpha, \dfrac{\lambda}{c})$ onde c constante positiva

- **QUI-QUADRADO** $X \sim \chi^2(n)$ **,** (n inteiro positivo).

$X \sim \chi^2(n) \Leftrightarrow X \sim \text{G}(n/2; 1/2)$; $E(X) = n$; $\text{Var}(X) = 2n$; $M(s) = (1 - 2s)^{-(n/2)}$

Propriedades:

- $X_i \sim \chi^2_{(n_i)}$ (i = 1, 2, ..., k) independentes $\Rightarrow \sum_{i=1}^{k} X_i \sim \chi^2_{(n)}$ com $n = \sum_{i=1}^{k} n_i$
- $X \sim \text{G}(n; \lambda) \Leftrightarrow 2\lambda X \sim \chi^2(2n)$
- $X_i \sim N(0,1)$, $(i = 1,2,...,n)$ independentes $\Rightarrow \sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$

- ***t*-"STUDENT"**

$T = \dfrac{U}{\sqrt{V/n}} \sim t(n)$ com $U \sim N(0{,}1)$ e $V \sim \chi^2(n)$ independentes

$E(T) = 0$; $\text{Var}(T) = \dfrac{n}{n-2}$ $(n > 2)$; Sendo $T \sim t(n) \Rightarrow \lim\limits_{n\to\infty} F_T(t \mid n) = \Phi(t)$

- ***F*-SNEDCOR**

$F = \dfrac{U/m}{V/n} \sim F(m,n)$ com $U \sim \chi^2(m)$, $V \sim \chi^2(n)$ (independentes)

Propriedades: . $X \sim F(m,n) \Rightarrow \dfrac{1}{X} \sim F(n,m)$. $T \sim t_{(n)} \Rightarrow T^2 \sim F(1,n)$

TEOREMA DO LIMITE CENTRAL E COROLÁRIOS

TLC: Sendo X_i iid com $E(X_i) = \mu$ e $\text{Var}(X_i) = \sigma^2$ $\Rightarrow \dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\,\sigma} = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \overset{a}{\sim} N(0{,}1)$

Corolário: Sendo $X_i \sim B(1;\theta)$, iid $\Rightarrow \dfrac{\sum_{i=1}^{n} X_i - n\theta}{\sqrt{n\theta(1-\theta)}} \overset{a}{\sim} N(0{,}1)$

Correcção de continuidade: $P(a \le X \le b) \approx \Phi\left(\dfrac{b+\frac{1}{2}-n\theta}{\sqrt{n\theta(1-\theta)}}\right) - \Phi\left(\dfrac{a-\frac{1}{2}-n\theta}{\sqrt{n\theta(1-\theta)}}\right)$, com a e b inteiros

Corolário: Sendo $X \sim \text{Po}(\lambda)$, quando $\lambda \to +\infty \Rightarrow \dfrac{X-\lambda}{\sqrt{\lambda}} \overset{a}{\sim} N(0{,}1)$

Correcção de continuidade: $P(a \le X \le b) \approx \Phi\left(\dfrac{b+\frac{1}{2}-\lambda}{\sqrt{\lambda}}\right) - \Phi\left(\dfrac{a-\frac{1}{2}-\lambda}{\sqrt{\lambda}}\right)$, com a e b inteiros