Stochastic Calculus - part 2

Master programme in Mathematical Finance

ISEG

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Definition

A s.p. $\{X_t; t \in T\}$ with values in $\mathbb{R}$ and where $T \subset \mathbb{R}$ is an interval, is said to be continuous in probability (or stochastically continuous) at $t \in T$ if, for all $\varepsilon > 0$,

$$\lim_{s \to t} P\left[|X_s - X_t| > \varepsilon\right] = 0.$$

Definition

Let $p \geq 1$. A s.p. $\{X_t; t \in T\}$ with values in $\mathbb{R}$ and where $T \subset \mathbb{R}$ is an interval, and such that $E\left[|X_t|^p\right] < \infty$, is said to be continuous in mean of order p at $t \in T$ if

$$\lim_{s \to t} E\left[|X_s - X_t|^p\right] = 0.$$

- The continuity in mean of order p implies the continuity in probability.
- The continuity in probability or in mean of order p does not imply the continuity of the trajectories of the process.

Example

The Poisson process $N = \{N_t, t \geq 0\}$ with intensity λ is a process with discontinuous trajectories. However it is continuous in mean of order 2 (or continuous in mean-square). Recall that $N_s - N_t \sim Poi(\lambda(s-t))$ and therefore (by the properties of the Poisson distribution)

$$\lim_{s \to t} E\left[|N_s - N_t|^2\right] = \lim_{s \to t}\left[\lambda(s-t) + (\lambda(s-t))^2\right] = 0.$$

- How to prove that a process has continuous trajectories?

Theorem

(Kolmogorov continuity criterion): Let $X = \{X_t; t \in T\}$ be a s.p. where T is a bounded interval and assume that exist $p > 0$ and $\alpha > 0$ such that

$$E\left[|X_t - X_s|^p\right] \leq C\,|t-s|^{1+\alpha}. \tag{1}$$

Then, exists a version of X with continuous trajectories.

- More precisely, Eq. (1) implies that for each $\varepsilon > 0$ exists a r.v. G_ε such that (with probability 1 or a.s.)

$$|X_t(\omega) - X_s(\omega)| \leq G_\varepsilon(\omega)\,|t-s|^{\frac{1+\alpha}{p}-\varepsilon} \tag{2}$$

  and $E\left[G_\varepsilon^p\right] < \infty$. That is, X has Hölder contínuous trajectories of order β for all $\beta < \frac{1+\alpha}{p}$.
- For a proof of this theorem, see Karatzas and Shreve, pages 53-54 (2nd edition).

Conditional probability

- Consider a probability space $(\Omega, \mathcal{F}, P)$ and let A and B be two events $A, B \in \mathcal{F}$ and $P(B) > 0$.
- Conditional probability of A given B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{3}$$

- The map $A \to P(A|B)$ defines a probability measure on the σ-algebra $\mathcal{F}$.
- Conditional expectation of X (integrable) given B:

$$E(X|B) = \frac{E[X\mathbf{1}_B]}{P(B)}. \tag{4}$$

Conditional expectation

- Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{B} \subset \mathcal{F}$ a σ-algebra.

Definition

(conditional expectation) The conditional expectation of an integrable r.v. X given $\mathcal{B}$ (or $E(X|\mathcal{B})$) is an integrable r.v. Z such that:

1. Z is $\mathcal{B}$-measurable.
2. For each $A \in \mathcal{B}$ we have

$$E(Z\mathbf{1}_A) = E(X\mathbf{1}_A) \tag{5}$$

- If X is integrable (i.e. $E[|X|] < \infty$) then $Z = E(X|\mathcal{B})$ exists and is unique (a.s.).

Definition

(generated σ-algebra): Let $\mathcal{C}$ be a class of subsets of Ω. Then, the smallest σ-algebra containing $\mathcal{C}$ is denoted by $\sigma(\mathcal{C})$ and is called the σ-algebra generated by $\mathcal{C}$.

Definition

(σ-algebra generated by X): Let X be a r.v. Then the σ-algebra $\{X^{-1}(B) : B \in \mathcal{B}_{\mathbb{R}}\}$ is said to be the σ-algebra generated by X. (By $\mathcal{B}_{\mathbb{R}}$ we denote the Borel σ-algebra in $\mathbb{R}$-generated by the open sets)

- Properties:

1.
$$E(aX + bY|\mathcal{B}) = aE(X|\mathcal{B}) + bE(Y|\mathcal{B}). \tag{6}$$

2.
$$E(E(X|\mathcal{B})) = E(X). \tag{7}$$

3. If X and the σ-algebra $\mathcal{B}$ are independent then:
$$E(X|\mathcal{B}) = E(X) \tag{8}$$

4. If X is $\mathcal{B}$-measurable (or if $\sigma(X) \subset \mathcal{B}$) then:
$$E(X|\mathcal{B}) = X. \tag{9}$$

5. If Y is $\mathcal{B}$-measurable (or if $\sigma(Y) \subset \mathcal{B}$) then
$$E(YX|\mathcal{B}) = YE(X|\mathcal{B}) \tag{10}$$

6. Given two σ-algebras $\mathcal{C} \subset \mathcal{B}$ then
$$E(E(X|\mathcal{B})|\mathcal{C}) = E(E(X|\mathcal{C})|\mathcal{B}) = E(X|\mathcal{C}) \tag{11}$$

7. Consider two r.v. X and Z such that Z is $\mathcal{B}$-measurable and X is independent of $\mathcal{B}$. Let $h(x,z)$ be a measurable function such that $h(X,Z)$ is an integrable r.v. Then
$$E(h(X,Z)|\mathcal{B}) = E(h(X,z))|_{z=Z}. \tag{12}$$

Note: first calculate $E(h(X,z))$ for a fixed value z of Z and then replace z by Z.

- Jensen inequality: If φ is a convex function such that $E\left[|\varphi(X)|\right] < \infty$, then

$$\varphi\left(E(X|\mathcal{B})\right) \leq E(\varphi(X)|\mathcal{B}). \tag{13}$$

- Particular case: If $E(|X|^p) < \infty$, $p \geq 1$,

$$|E(X|\mathcal{B})|^p \leq E(|X|^p|\mathcal{B}).$$

As a consequence, if $p \geq 1$,

$$E\left[|E(X|\mathcal{B})|^p\right] \leq E(|X|^p). \tag{14}$$

- We can define for $C \in \mathcal{F}$,

$$P(C|\mathcal{B}) = E(\mathbf{1}_C|\mathcal{B}).$$

Conditional expectation

- The set of all squared integrable r.v. ($E\left[X^2\right] < \infty$) - denoted by $L^2(\Omega, \mathcal{F}, P)$ - is a Hilbert space with the inner product

$$\langle X, Y\rangle = E\left[XY\right].$$

The set $L^2(\Omega, \mathcal{B}, P)$ is a subspace of $L^2(\Omega, \mathcal{F}, P)$.

- Given $X \in L^2(\Omega, \mathcal{F}, P)$, we have that $E(X|\mathcal{B})$ is the orthogonal projection of X in the subspace $L^2(\Omega, \mathcal{B}, P)$ and minimizes the mean-square distance from X to $L^2(\Omega, \mathcal{B}, P)$, in the sense that

$$E\left[(X - E(X|\mathcal{B}))^2\right] = \min_{Y \in L^2(\Omega,\mathcal{B},P)} E\left[(X-Y)^2\right] \tag{15}$$

Examples and exercises

Example

Let X be a uniform r.v. with values on $(0, 1]$. Let $A = \left(0, \frac{1}{4}\right]$. Calculate $E[X]$ and $E[X|A]$.

$$E[X] = \int_0^1 xf(x)\,dx = \int_0^1 xdx = \frac{1}{2}.$$

$$E[X|A] = \frac{E(X\mathbf{1}_A)}{P(A)} = \frac{\int_0^{1/4} xdx}{1/4} = \frac{1}{8}.$$

- Exercise: Prove that if X and the σ-algebra $\mathcal{B}$ are independent then $E(X|\mathcal{B}) = E(X)$
  Solution: X and $\mathbf{1}_A$ are independent if $A \in \mathcal{B}$ and

$$E[X\mathbf{1}_A] = E[X]\,E[\mathbf{1}_A] = E[X]\,P(A) = E[E[X]\,\mathbf{1}_A]$$

  and, by definition of conditional expectation, $E(X|\mathcal{B}) = E(X)$.

- Exercise: Prove that if Y is $\mathcal{B}$-measurable then

$$E(YX|\mathcal{B}) = YE(X|\mathcal{B}).$$

Solution sketch: If $Y = \mathbf{1}_A$ with $A, B \in \mathcal{B}$ then, by definition of conditional expectation,

$$\begin{aligned} E\left[\mathbf{1}_A E(X|\mathcal{B})\mathbf{1}_B\right] &= E\left[\mathbf{1}_{A\cap B} E(X|\mathcal{B})\right] \\ &= E\left[X\mathbf{1}_{A\cap B}\right] = E\left[\mathbf{1}_B \mathbf{1}_A X\right]. \end{aligned}$$

Therefore $\mathbf{1}_A E(X|\mathcal{B}) = E\left[\mathbf{1}_A X|\mathcal{B}\right]$. In a similar whay, we obtain the result for $Y = \sum_{j=1}^m a_j \mathbf{1}_{A_j}$ (a simple r.v.). In the general case, we prove the result, approximating Y by a sequence of simple (and $\mathcal{B}$-measurable) random variables.

- Exercise: Given $X \in L^2(\Omega, \mathcal{F}, P)$, show that $E(X|\mathcal{B})$ is the orthogonal projection of X in subspace $L^2(\Omega, \mathcal{B}, P)$ and that

$$E\left[(X - E(X|\mathcal{B}))^2\right] = \min_{Y \in L^2(\Omega,\mathcal{B},P)} E\left[(X - Y)^2\right].$$

Solution:(1) $E(X|\mathcal{B}) \in L^2(\Omega, \mathcal{B}, P)$ since is $\mathcal{B}$-measurable and by (14) we have that

$$E\left[|E(X|\mathcal{B})|^2\right] \leq E(|X|^2) < \infty.$$

(2) If $Z \in L^2(\Omega, \mathcal{B}, P)$ then, by properties 5 and 2 of cond. expect.:

$$\begin{aligned} E\left[(X - E(X|\mathcal{B}))\, Z\right] &= E\left[XZ\right] - E\left[E(X|\mathcal{B})Z\right] \\ &= E\left[XZ\right] - E\left[E(XZ|\mathcal{B})\right] \\ &= 0 \end{aligned}$$

and therefore $(X - E(X|\mathcal{B}))$ is orthogonal to $L^2(\Omega, \mathcal{B}, P)$.

(3) Since

$$E\left[(X-Y)^2\right] = E\left[(X-E(X|\mathcal{B}))^2\right] + E\left[(E(X|\mathcal{B})-Y)^2\right]$$

we have that $E\left[(X-Y)^2\right] \geq E\left[(X-E(X|\mathcal{B}))^2\right]$ .Hence

$$E\left[(X-E(X|\mathcal{B}))^2\right] = \min_{Y\in L^2(\Omega,\mathcal{B},P)} E\left[(X-Y)^2\right].$$

- Exercise: Prove properties 2, 4 and 6 of the conditional expectation.