Stochastic Calculus - part 3

ISEG

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Martingales in discrete time

- Consider a probab. space $(\Omega, \mathcal{F}, P)$ and a sequence of σ-algebras $\{\mathcal{F}_n, n \geq 0\}$ such that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}$$

  The sequence $\{\mathcal{F}_n, n \geq 0\}$ is called a filtration.
- Filtration $\approx$ information flow.

Definition

A s.p. $M = \{M_n; n \geq 0\}$ in discrete time is a martingale with respect to $\{\mathcal{F}_n, n \geq 0\}$ if

1. For each n, M_n is a $\mathcal{F}_n$-measurable r.v. (i.e., M is a s.p. adapted to the filtration $\{\mathcal{F}_n, n \geq 0\}$).
2. For each n, $E[|M_n|] < \infty$.
3. For each n, we have
$$E[M_{n+1}|\mathcal{F}_n] = M_n.$$

- The s.p. $M = \{M_n; n \geq 0\}$ is a supermartingale (resp. submartingale) if satisfies conditions 1 and 2 of the previous difinitions and condition 3 is replaced by $E[M_{n+1}|\mathcal{F}_n] \leq M_n$ (resp. $E[M_{n+1}|\mathcal{F}_n] \geq M_n$).
- Cond. (3) $\Longrightarrow E[M_n] = E[M_0]$ for all $n \geq 1$. (Homework: prove this
- Cond (3) $\Longleftrightarrow E[\Delta M_n|\mathcal{F}_{n-1}] = 0$ for all $n \geq 1$, where $\Delta M_n := M_n - M_{n-1}$.
- Cond. 3 $\approx$ "Given the information $\mathcal{F}_n$, the best estimate for M_{n+1} is M_n."

Example

(Random walk): Let $\{Z_n; n \geq 0\}$ be a seq. of integrable and independent r.v. with zero expected value. Let $M = \{M_n; n \geq 0\}$ with

$$M_n = Z_0 + Z_1 + \cdots + Z_n.$$

Consider the natural filtration generated by $\{Z_n; n \geq 0\}$, i.e.,

$$\mathcal{F}_n := \sigma\{Z_0, Z_1, \ldots, Z_n\}.$$

Since $M_0, M_1, \ldots, M_n$ and $Z_0, Z_1, \ldots, Z_n$ generate the same information, they generate the same σ-algebra $\mathcal{F}_n$. Let us prove that M is a martingale:
1. M is adapted to the filtration $\{\mathcal{F}_n, n \geq 0\}$ since M_n is $\mathcal{F}_n$-measurable (M_n is one of the r.v. that generates $\mathcal{F}_n$).
2. $E[|M_n|] < \infty$, because all the r.v. Z_n are integrable (i.e. $E[|Z_n|] < \infty$ for all n).

Example

3. By the basic properties of conditional expectation:

$$\begin{aligned} E\left[M_{n+1}|\mathcal{F}_n\right] &= E\left[M_n + Z_{n+1}|\mathcal{F}_n\right] \\ &= M_n + E\left[Z_{n+1}|\mathcal{F}_n\right] \\ &= M_n + E\left[Z_{n+1}\right] \\ &= M_n. \end{aligned}$$

- Note: the σ-algebra $\sigma\left(X_1, X_2, \ldots, X_n\right)$ generated by the r.v. $\left(X_1, X_2, \ldots, X_n\right)$ contains all the "essential information" about the "structure" of the random vector $\left(X_1, X_2, \ldots, X_n\right)$ (as a map of $\omega \in \Omega$).

Lemma

Let $M = \{M_n; n \geq 0\}$ be a martingale with respect to the filtration $\{\mathcal{G}_n, n \geq 0\}$ and $\mathcal{F}_n = \sigma\{M_0, M_1, \ldots, M_n\} \subset \mathcal{G}_n$ is the natural filtration generated by M. Then M is a martingale with respect to $\{\mathcal{F}_n, n \geq 0\}$.

Proof.

By property 6 of the conditional expectation and by the martingale property:

$$\begin{aligned} E\left[M_{n+1}|\mathcal{F}_n\right] &= E\left[E\left[M_{n+1}|\mathcal{G}_n\right]|\mathcal{F}_n\right] \\ &= E\left[M_n|\mathcal{F}_n\right] \\ &= M_n. \end{aligned}$$

□

- Some martingale properties:
  1. Let $M = \{M_n; n \geq 0\}$ be a $\{\mathcal{F}_n\}$-martingale. Then, for $m \geq n$:
  $$E[M_m|\mathcal{F}_n] = M_n. \quad \text{(Exerc.: prove this)}$$
  2. $\{M_n; n \geq 0\}$ is a submartingale iff $\{-M_n; n \geq 0\}$ is a supermartingale.
  3. If $\{M_n; n \geq 0\}$ is a martingale and φ is a convex function such that $E[|\varphi(M_n)|] < \infty \ \ \forall n \geq 0$, then $\{\varphi(M_n), n \geq 0\}$ is a submartingale.
- Poperty 3. is a consequence of the Jensen inequality and has the corollary: if $\{M_n; n \geq 0\}$ is a martingale and $E[|M_n|^p] < \infty \ \ \forall n \geq 0$ for some $p \geq 1$, then $\{|M_n|^p, n \geq 0\}$ is a submartingale.

Let $\{\mathcal{F}_n, n \geq 0\}$ be a filtration on $(\Omega, \mathcal{F}, P)$.

Definition

The s.p. $\{H_n, n \geq 1\}$ is called a predictable process if H_n is $\mathcal{F}_{n-1}$-measurable (i.e., if H_n is "known" at time $n-1$).

Definition

Given a $\{\mathcal{F}_n\}$-martingale $M = \{M_n; n \geq 0\}$ and a predictable process $\{H_n, n \geq 1\}$, the process $\{(H \cdot M)_n, n \geq 1\}$, defined by

$$(H \cdot M)_n = M_0 + \sum_{j=1}^{n} H_j \Delta M_j$$

is called the martingale transform of M by $\{H_n, n \geq 1\}$ or the discrete stochastic integral of H with respect to M.

- The martingale transform by a predictable process is the discrete version of the stochastic integral:

$$(H \cdot M)_n - M_0 = \sum_{j=1}^{n} H_j \Delta M_j \approx \int_0^n H_s dM_s.$$

Theorem

If $M = \{M_n; n \geq 0\}$ is a martingale and $\{H_n, n \geq 0\}$ is a predictable process with bounded random variables, then the martingale transform (or discrete stochastic integral) $\{(H \cdot M)_n, n \geq 1\}$ is a martingale.

Proof.

1. $(H \cdot M)_n$ is $\{\mathcal{F}_n\}$-measurable since $\sum_{j=1}^{n} H_j \Delta M_j$ is $\mathcal{F}_n$- measurable.
2. $(H \cdot M)_n$ is integrable because the r.v. M_n are integrable and the r.v. H_n are bounded.
3. By the properties of conditional expectation:

$$\begin{aligned} E\left[(H \cdot M)_{n+1} - (H \cdot M)_n | \mathcal{F}_n\right] &= E\left[H_{n+1}\left(M_{n+1} - M_n\right) | \mathcal{F}_n\right] \\ &= H_{n+1} E\left[M_{n+1} - M_n | \mathcal{F}_n\right] \\ &= 0. \end{aligned}$$

□

Example

Game and betting system:Let H_n be the amount of the bet by a player on time n; $\Delta M_n = M_n - M_{n-1}$ is the amount won on time n; M_n: total amount accumulated by player at time n; $(H \cdot M)_n$: is the total amount accumulated by player at time n if he uses the betting system $\{H_n, n \geq 1\}$. If $\{M_n; n \geq 0\}$ is a martingale we say that the game is fair. Then $(H \cdot M)_n$ is also a martingale - that is, the game remains a "fair game", for any betting system used by the player such that $\{H_n, n \geq 0\}$ satisfies the conditions of the previous theorem.

Example

(doubling bet system): Assume that $M_n = M_0 + Z_1 + \cdots + Z_n$, where $\{Z_n; n \geq 1\}$ are indep. r.v. that represent "heads" $(+1)$ or "tails" (-1) in a flipping coin: $P(Z_i = 1) = P(Z_i = -1) = \frac{1}{2}$. The player initially bets one Euro and he doubles his bet if the result is "tails" (-1) (doubles his bet when he loses) and ends the game whenever he gets "heads" $(+1)$. That is, $H_1 = 1$, $H_n = 2H_{n-1}$ if $Z_{n-1} = -1$ and $H_n = 0$ if $Z_{n-1} = +1$. If the player loses k times and wins at time $k+1$, he gets:

$$(H \cdot M)_k = -1 - 2 - 4 - \cdots - 2^{k-1} + 2^k = 1.$$

It seems that the strategy is always a winning strategy. But be carefull, in order to be a winning strategy (with probability 1) the player needs unbounded resources (infinite amount of money) - unbounded r.v. for the betting system - and unbounded time.

Application to Finance

Example

Let $S_n := \left\{S_n^0, S_n^1, n \geq 1\right\}$ be adapted processes that represent the price of two assets. $S_n^0 = (1+r)^n$ is the price of the riskless asset (riskless bond), where r is the interest rate (S_n^0 is deterministic). A portfolio is a predictable process $\phi_n := \left\{\phi_n^0, \phi_n^1, n \geq 1\right\}$ and the value of the portfolio at time n is

$$V_n = \phi_n^0 S_n^0 + \phi_n^1 S_n^1 = \phi_n \cdot S_n$$

The portfolio is said to be self-financing if, for all n,

$$V_n = V_0 + \sum_{j=1}^{n} \phi_j \Delta S_j.$$

This condition is equivalent to

$$\phi_n \cdot S_n = \phi_{n+1} \cdot S_n$$

Example

Define the discounted prices

$$\widetilde{S}_n = (1+r)^{-n} S_n = \left(1, (1+r)^{-n} S_n^1\right)$$

Clearly, we have

$$\widetilde{V}_n = (1+r)^{-n} V_n = \phi_n \cdot \widetilde{S}_n,$$

$$\phi_n \cdot \widetilde{S}_n = \phi_{n+1} \cdot \widetilde{S}_n,$$

$$\widetilde{V}_n = V_0 + \sum_{j=1}^{n} \phi_j \Delta \widetilde{S}_j$$

$\widetilde{V}_n = \left(\phi_n^1 \cdot \widetilde{S}^1\right)_n$ is the martingale transform of $\left\{\widetilde{S}_n^1\right\}$ by the process $\left\{\phi_n^1\right\}$. Then, if $\left\{\widetilde{S}_n^1\right\}$ is a martingale and $\left\{\phi_n^1\right\}$ is a bounded sequence (bounded r.v.), then $\left\{\widetilde{V}_n\right\}$ is also a martingale (by the previous theorem).

Example

A probability measure Q equivalent to P is a risk neutral probability measure if on the proab. space $(\Omega, \mathcal{F}, Q)$, the process $\left\{\widetilde{S}_n^1\right\}$ is a $\{\mathcal{F}_n\}$-martingale. In that case, if $\{\phi_n^1\}$ is bounded, $\left\{\widetilde{V}_n\right\}$ is also a martingale.
In the binomial model, we assume that the r.v.

$$T_n = \frac{S_n}{S_{n-1}}$$

are independent and can have the values $1+a$ and $1+b$ with probabilities p and $1-p$, resp., with $a < r < b$.

Example

Let us find p (or the probab. measure Q) in such a way that $\left\{\widetilde{S}_n^1\right\}$ is a martingale.

$$\begin{aligned} E\left[\widetilde{S}_{n+1}|\mathcal{F}_n\right] &= (1+r)^{-n-1} E\left[S_n T_{n+1}|\mathcal{F}_n\right] \\ &= \widetilde{S}_n (1+r)^{-1} E\left[T_{n+1}|\mathcal{F}_n\right] \\ &= \widetilde{S}_n (1+r)^{-1} E\left[T_{n+1}\right] \end{aligned}$$

Therefore, $\left\{\widetilde{S}_n^1\right\}$ is a martingale iff $E\left[T_{n+1}\right] = (1+r)$. That is,

$$E\left[T_{n+1}\right] = p(1+a) + (1-p)(1+b) = 1+r$$

and therefore

$$p = \frac{b-r}{b-a}.$$

Example

Consider now a r.v.. H which is $\{\mathcal{F}_N\}$-measurable and represents the payoff of an option or financial derivative on the asset 1 with maturity at time N. For example, a "call" option has payoff $H = (S_T - K)^+$. The derivative is said to be replicable if exists a self-financing portfolio such that

$$V_N = H.$$

The price of the derivative is the value of this portfolio. Since $\left\{\widetilde{V}_n\right\}$ is a Q-martingale, we have

$$\begin{aligned} V_n &= (1+r)^n \widetilde{V}_n = (1+r)^n E_Q\left[\widetilde{V}_N | \mathcal{F}_n\right] \\ &= (1+r)^{-(N-n)} E_Q\left[H|\mathcal{F}_n\right] \end{aligned}$$

If $n = 0$, we have $\mathcal{F}_0 = \{\Omega, \varnothing\}$ and

$$V_0 = (1+r)^{-N} E_Q[H].$$

Martingales in continuous time

- Martingales in continuous time are defined in analogous way as in discrete time and most properties remain in continuous time.
- Probab. space $(\Omega, \mathcal{F}, P)$ and a family of σ-algebras $\{\mathcal{F}_t, t \geq 0\}$ such that

  $$\mathcal{F}_s \subset \mathcal{F}_t, \quad 0 \leq s \leq t.$$

  The sequence $\{\mathcal{F}_t, t \geq 0\}$ is called a filtration.
- Let $\mathcal{F}_t^X$ be a σ-algebra generated by process X on the interval $[0, t]$, i.e. $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$. Then $\mathcal{F}_t^X$ is the "information generated by X on interval $[0, t]$".
- $A \in \mathcal{F}_t^X$ means that it is possible to decide if event A has occured or not, based on the observation of trajectories of X on $[0, t]$.
- Example: If $A = \{\omega : X(5) > 1\}$ then $A \in \mathcal{F}_5^X$ but $A \notin \mathcal{F}_4^X$.

Definition

A s.p. $M = \{M_t; t \geq 0\}$ is said to be a martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$ if:

1. For each $t \geq 0$, M_t is a r.v. which is $\mathcal{F}_t$-measurable (i.e., M is a s.p. adapted to $\{\mathcal{F}_t, t \geq 0\}$).
2. For each $t \geq 0$, $E[|M_t|] < \infty$.
3. For each $s \leq t$,
$$E[M_t|\mathcal{F}_s] = M_s.$$

- Cond (3) $\Longleftrightarrow E[M_t - M_s|\mathcal{F}_s] = 0$.
- If $t \in [0, T]$ then $M_t = E[M_T|\mathcal{F}_t]$.
- The definitons of supermartingale and submartingale are similar to the definitions for discrete time.
- As in the discrete time case, Cond. (3) $\Longrightarrow E[M_t] = E[M_0]$ for all t.

We have the following generalization of the Chebyshev inequality (analogous to the discrete time version).

Theorem

(Maximal inequality (or martingale inequality) of Doob): If $M = \{M_t; t \geq 0\}$ *is a martingale with continuous trajectories then, for all* $p \geq 1$, $T \geq 0$ *and* $\lambda > 0$,

$$P\left[\sup_{0 \leq t \leq T} |M_t| \geq \lambda\right] \leq \frac{1}{\lambda^p}\left[E\,|M_T|^p\right]$$

For a proof of this theorem in discrete time (based on the optional stopping theorem) see the lecture notes "Stochastic Calculus" by D. Nualart.