Stochastic Calculus - part 6

ISEG

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The stochastic integral as a process

- Consider a stochastic process $u \in L^2_{a,T}$. Then, for each $t \in [0, T]$, the process $u\mathbf{1}_{[0,t]}$ also belongs to $L^2_{a,T}$ and we can define the indefinite stochastic integral:

$$\int_0^t u_s \mathrm{d}B_s := \int_0^T u_s \mathbf{1}_{[0,t]}(s)\, \mathrm{d}B_s.$$

- The stochastic process $\left\{ \int_0^t u_s \mathrm{d}B_s,\ 0 \leq t \leq T \right\}$ is the indefinite stochastic integral of u with respect to B.

- Main properties of the indefinite integral:

1. Additivity: For any $a \leq b \leq c$, we have:

$$\int_a^b u_s \mathrm{d}B_s + \int_b^c u_s \mathrm{d}B_s = \int_a^c u_s \mathrm{d}B_s.$$

2. Factorization: For $a < b$ and $A \in \mathcal{F}_a$, we have:

$$\int_a^b \mathbf{1}_A u_s \mathrm{d}B_s = \mathbf{1}_A \int_a^b u_s \mathrm{d}B_s.$$

   This property remains valid if we replace $\mathbf{1}_A$ by any bounded random variable which is also $\mathcal{F}_a$-measurable.

3. Martingale property: If $u \in L^2_{a,T}$ then the indefinite stochastic integral $M_t = \int_0^t u_s \mathrm{d}B_s$ is a $\{\mathcal{F}_t\}$-martingale.

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4. Continuity: If $u \in L^2_{a,T}$ then the indefinite stochastic integral $M_t = \int_0^t u_s \mathrm{d}B_s$ has a version with continuous trajectories.
5. Maximal inequality for the indefinite stochastic integral: If $u \in L^2_{a,T}$ and $M_t = \int_0^t u_s \mathrm{d}B_s$, then, for any $\lambda > 0$, we have

$$P\left[\sup_{0 \leq t \leq T} |M_t| > \lambda\right] \leq \frac{1}{\lambda^2} E\left[\int_0^T u_t^2 \mathrm{d}t\right].$$

- Proof of 1: Exercise - (TPC).
- Proof of 3:
  Let $u^{(n)}$ be a sequence of simple processes such that

$$\lim_{n\to\infty} E\left[\int_0^T \left|u_t - u_t^{(n)}\right|^2 dt\right] = 0.$$

Let $M_n(t) = \int_0^t u_s^{(n)} \mathrm{d}B_s$. and let ϕ_j be the value of $u^{(n)}$ in $(t_{j-1}, t_j]$, with $j = 1, \ldots, n$.

If $s \leq t_k \leq t_{m-1} \leq t$, then:

$$E[M_n(t) - M_n(s) | \mathcal{F}_s]$$
$$= E\left[\phi_k(B_{t_k} - B_s) + \sum_{j=k+1}^{m-1} \phi_j \Delta B_j + \phi_m(B_t - B_{t_{m-1}}) | \mathcal{F}_s\right],$$

and by the properties of conditional expectation, we have:

$$= E[\phi_k(B_{t_k} - B_s) | \mathcal{F}_s] + \sum_{j=k+1}^{m-1} E[E[\phi_j \Delta B_j | \mathcal{F}_{j-1}] | \mathcal{F}_s] +$$
$$+ E[E[\phi_m(B_t - B_{t_{m-1}}) | \mathcal{F}_{t_{m-1}}] | \mathcal{F}_s].$$

$$= \phi_k E\left[B_{t_k} - B_s | \mathcal{F}_s\right] + \sum_{j=k+1}^{m-1} E\left[\phi_j E\left[\Delta B_j | \mathcal{F}_{j-1}\right] | \mathcal{F}_s\right] +$$
$$+ E\left[\phi_m E\left[B_t - B_{t_{m-1}} | \mathcal{F}_{t_{m-1}}\right] | \mathcal{F}_s\right]$$

and using the independence of Brownian motion increments, we get

$$= 0.$$

The mean square convergence implies mean square convergence of the conditional expectation, and therefore we have

$$E\left[M(t) - M(s) | \mathcal{F}_s\right] = 0$$

and the indefinite stochastic integral is a martingale.

- Proof of 4: $M_n(t)$ has clearly continuous trajectories, since it is the integral of a simple process (Exercíse: Prove this statement). Then, by the Doob maximal inequality applied to $M_n - M_m$, with $p = 2$, we obtain:

$$P\left[\sup_{0 \le t \le T} |M_n(t) - M_m(t)| > \lambda\right] \le \frac{1}{\lambda^2} E\left[|M_n(T) - M_m(T)|^2\right]$$

$$= \frac{1}{\lambda^2} E\left[\left(\int_0^T \left(u_t^{(n)} - u_t^{(m)}\right) dB_t\right)^2\right]$$

$$= \frac{1}{\lambda^2} E\left[\int_0^T \left|u_t^{(n)} - u^{(m)}\right|^2 dt\right] . \overset{n,m\to\infty}{\longrightarrow} 0,$$

where we used the Itô isometry.

We can therefore choose a subsequence n_k, $k = 1, 2, \ldots$, such that

$$P\left[\sup_{0\leq t\leq T}\left|M_{n_{k+1}}(t) - M_{n_k}(t)\right| > 2^{-k}\right] \leq 2^{-k}.$$

The events:

$$A_k := \left\{\sup_{0\leq t\leq T}\left|M_{n_{k+1}}(t) - M_{n_k}(t)\right| > 2^{-k}\right\}$$

satisfy

$$\sum_{k=1}^{\infty} P(A_k) < \infty.$$

Therefore, by the Borel-Cantelli Lemma, we have that $P(\limsup_k A_k) = 0$ or

$$P\left[\sup_{0\leq t\leq T}\left|M_{n_{k+1}}(t) - M_{n_k}(t)\right| > 2^{-k} \text{ for infinite values } k\right] = 0.$$

Therefore, for almost all $\omega \in \Omega$, exists a $k_1(\omega)$ such that

$$\sup_{0 \leq t \leq T} \left| M_{n_{k+1}}(t) - M_{n_k}(t) \right| \leq 2^{-k} \text{ for } k \geq k_1(\omega).$$

Hence, $M_{n_k}(t, \omega)$ is uniformly convergent on $[0, T]$ a.s.and therefore the limit, which we denote by $J_t(\omega)$, is a continuous function of t. Finally, since $M_{n_k}(t.\cdot) \rightarrow M_t(\cdot)$ in mean square (or in $L^2(\Omega)$) for all t,then we must have

$$M_t = J_t \quad \text{a.s. and for all } t \in [0, T],$$

and the indefinite stochastic integral has a continuous version.

Quadratic variation of the indefinite stochastic integral

- Let $u \in L^2_{a,T}$. Then

$$\sum_{j=1}^{n} \left(\int_{t_{j-1}}^{t_j} u_s dB_s \right)^2 \overset{L^1(\Omega)}{\longrightarrow} \int_0^t u_s^2 ds,$$

  when $n \rightarrow \infty$ and with $t_j := \frac{jt}{n}$.

Stochastic integral extension

- One can replace $\{\mathcal{F}_t\}$ (filtration generated by the Brownian motion) by a larger filtration $\mathcal{H}_t$ such that the Brownian motion B_t is a $\mathcal{H}_t$-martingale.
- We can replace condition 2) $E\left[\int_0^T u_t^2 dt\right] < \infty$ in the definition of $L^2_{a,T}$ by the (weaker) condition:
  2') $P\left[\int_0^T u_t^2 dt < \infty\right] = 1$.
- Let $L_{a,T}$ be the space of adapted and measurable processes u that satisfy condition 2'). This space is larger than $L^2_{a,T}$. The stochastic integral can be defined for processes $u \in L_{a,T}$ but, in this case, the stochastic integral may fail to have a mean value of zero and to satisfy the Itô isometry.

Exercises:

- Exercise: Prove direcly, by using the definition of stochastic integral, that

  $$\int_0^t s dB_s = tB_t - \int_0^t B_s ds.$$

  Sugestion: Note that

  $$\sum_j \Delta\left(s_j B_j\right) = \sum_j s_j \Delta B_j + \sum_j B_{j+1} \Delta s_j.$$

- Exercise: Consider a deterministic function g such that $\int_0^T g\left(s\right)^2 ds < \infty$. Show that the stochastic integral $\int_0^T g\left(s\right) dB_s$ is a Gaussian random variable and calculate its mean and variance.