ISEG - Lisbon School of Economics and Management
2017/2018

Statistics I
Problem set 3 - Multivariate Random Variables
(version: 20/03/2018)

1. If the values of the joint probability function of X and Y are as shown in the table

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

   (a) find:
      i. $P(X = 1, Y = 2)$;
      ii. $P(X = 0, 1 \leq Y < 3)$;
      iii. $P(X + Y \leq 1)$;
      iv. $P(X > Y)$.

   (b) find the following values of the joint cumulative distribution function of the two random variables:
      i. $F(1.2, 0.9)$;
      ii. $F(-3, 1.5)$;
      iii. $F(2, 0)$;
      iv. $F(4, 2.7)$.

2. If the joint probability function of X and Y is given by

$$f(x, y) = c(x^2 + y^2)$$

   for $x = -1, 0, 1, 3$; $y = -1, 2, 3$.

   (a) find the value of c.

   (b) find:
      i. $P(X \leq 1, Y > 2)$;
      ii. $P(X = 0, Y \leq 2)$;
      iii. $P(X + Y > 2)$.

3. Show that there is no value of k for which

$$f(x, y) = ky(2y - x),$$

   for $x = 0, 3$; $y = 0, 1, 2$ can serve as the joint probability function of two random variables.

4. If the joint probability distribution of X and Y is given by

$$f(x,y) = \frac{1}{30}(x+y)$$

for $x = 0,1,2,3$; $y = 0,1,2$ construct a table showing the values of the joint cumulative distribution function of the two random variables at the 12 points $(0,0)$, $(0,1)$, . . . , $(3,2)$.

5. Determine k so that

$$f(x,y) = \begin{cases} kx\,(x-y) & \text{, for } 0 < x < 1, \ -x < y < x \\ 0 & \text{, elsewhere} \end{cases}$$

can serve as a joint probability density function.

6. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 24xy & \text{, for } 0 < x < 1, \ 0 < y < 1, \ x+y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

find $P(X+Y < 1/2)$.

7. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 2 & \text{, for } x > 0, y > 0, \ x+y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

   (a) find

      i. $P(X \leq 1/2, Y \leq 1/2)$;

      ii. $P(X+Y > 2/3)$;

      iii. $P(X > 2Y)$.

   (b) find an expression for the values of the joint cumulative distribution function of X and Y for for $x > 0, y > 0, \ x+y < 1$.

8. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{x} & 0 < x < 1, \text{for } 0 < y < x, \\ 0 & \text{, elsewhere} \end{cases}$$

find the probability that the sum of the values of X and Y will exceed $1/2$.

9. If the joint cumulative distribution function of X and Y is given by

$$F(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{, for } x > 0, \ y > 0 \\ 0 & \text{, elsewhere} \end{cases}$$

(a) find the joint probability density of the two random variables X and Y.

(b) find $P(1 < X \leq 2, 1 < Y \leq 2)$.

10. If the joint cumulative distribution function of X and Y is given by

$$F(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-x-y} & \text{, for } x > 0,\ y > 0 \\ 0 & \text{, elsewhere} \end{cases}$$

(a) find the joint probability density of the two random variables X and Y.

(b) find $P(X + Y > 3)$.

11. Two random variables have the following joint distribution:

	$X = 1$	$X = 2$	$X = 3$
$Y = 1$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{9}$
$Y = 2$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{2}{9}$
$Y = 3$	$\frac{1}{9}$	$\frac{2}{9}$	0

(a) What is $P(Y = 2)$?

(b) What is $P(Y = 2 | X = 2)$?

(c) Are X and Y independent?

12. Given the values of the joint probability distribution of X and Y shown in the table

	$X = -1$	$X = 1$
$Y = -1$	$\frac{1}{8}$	$\frac{1}{2}$
$Y = 0$	0	$\frac{1}{4}$
$Y = 1$	$\frac{1}{8}$	0

find

(a) the marginal probability function of X;

(b) the marginal probability function of Y;

(c) the conditional probability function of X given $Y = -1$.

13. With reference to question 1, find

(a) the marginal probability function of X;

(b) the marginal probability function of Y;

(c) the conditional probability function of X given $Y = 1$;

(d) the conditional probability function of Y given $X = 0$.

14. Check whether X and Y are independent if their joint probability function is given by

(a) $f(x, y) = 1/4$ for $x = -1$ and $y = -1$; $x = -1$ and $y = 1$; $x = 1$ and $y = -1$; and $x = 1$ and $y = 1$;

(b) $f(x, y) = 1/3$ for $x = 0$ and $y = 0$; $x = 0$ and $y = 1$; and $x = 1$ and $y = 1$.

15. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{, for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{, elsewhere} \end{cases}$$

find

(a) the marginal density of X;

(b) the conditional density of Y given $X = 1/4$

(c) the marginal density of Y;

(d) the conditional density of X given $Y = 1$.

16. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 24y(1-x-y) & \text{, for } x > 0, y > 0, x + y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

(a) find the marginal density of X;

(b) find the marginal density of Y;

(c) determine whether the two random variables are independent.

17. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{y} & \text{, for } 0 < x < y,\ 0 < y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

(a) find the marginal density of X;

(b) find the marginal density of Y.

(c) determine whether the two random variables are independent.

18. With reference to question 9 find the marginal cumulative distribution function of X.

19. Let $X \sim U(0, 2)$, $Y \sim U(0, 3)$ and X and Y are independent random variables, find the joint probability density function of X and Y;

20. Suppose that P, the price of a certain commodity (in dollars), and S, its total sales (in $10,000$ units), are random variables whose joint probability distribution can be approximated closely with the joint probability density

$$f(p,s) = \begin{cases} 5pe^{-ps} & \text{, for } 0.2 < p < 0.4, s > 0 \\ 0 & \text{, elsewhere} \end{cases}$$

(a) find the probabilities that

  i. the price will be less than 30 cents and sales will exceed $20,000$ units;

  ii. the price will be between 25 cents and 30 cents and sales will be less than $10,000$ units.

(b) find

  i. the marginal density of P;

  ii. the conditional density of S given $P = p$;

  iii. the probability that sales will be less than $30,000$ units when $p = 25$ cents.

21. In a company, if X is the proportion of persons who will respond to the first kind of mail-order solicitation, Y is the proportion of persons who will respond to the second kind of mail-order and the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{2}{5}(x+4y) & \text{, for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

find the probabilities that

(a) at least 30 percent will respond to the first kind of mail-order solicitation;

(b) at most 50 percent will respond to the second kind of mail-order solicitation given that there has been a 20 percent response to the first kind of mail-order solicitation.

22. If X is the amount of money (in dollars) that a salesperson spends on gasoline during a day and Y is the corresponding amount of money (in dollars) for which he or she is reimbursed, the joint density of these two random variables is given by

$$f(x,y) = \begin{cases} \frac{1}{25}\left(\frac{20-x}{x}\right) & \text{, for } 10 < x < 20, \frac{x}{2} < y < x \\ 0 & \text{, elsewhere} \end{cases}$$

find

(a) the marginal density of X;

(b) the conditional density of Y given $X = 12$;

(c) the probability that the salesperson will be reimbursed at least \$8 when spending \$12.