ISEG - Lisbon School of Economics and Management
2017/2018

Statistics I
Problem set 2 - Random Variables
(version: 1/04/2018)

1. For each of the following, determine whether the given values can serve as the values of a probability function of a random variable with the range $x = 1$, 2, 3, and 4:

   (a) $f(1) = 0.25$, $f(2) = 0.75$, $f(3) = 0.25$, and $f(4) = -0.25$;
   (b) $f(1) = 0.15$, $f(2) = 0.27$, $f(3) = 0.29$, and $f(4) = 0.29$;
   (c) $f(1) = 1/19$, $f(2) = 10/19$, $f(3) = 2/19$, and $f(4) = 5/19$

2. Verify that $f(x) = 2x/[k(k+1)]$ for $x = 1, 2, 3, ..., k$ can serve as the probability function of a random variable with the given range.

3. For each of the following, determine c so that the function can serve as the probability function of a random variable with the given range:

   (a) $f(x) = cx$, for $x = 1, 2, 3, 4, 5$;
   (b) $f(x) = c\binom{5}{x}$, for $x = 0, 1, 2, 3, 4, 5$, [**Hint:** $2^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}$];
   (c) $f(x) = cx^2$, for $x = 1, 2, 3, ..., k$;
   (d) $f(x) = c\,(1/4)^x$ for $x = 1, 2, 3, ...$

4. For what values of k can $f(x) = (1-k)k^x$ serve as the values of the probability function of a random variable with the countably infinite range $x = 0, 1, 2, ...$?

5. Show that there are no values of c such that $f(x) = c/x$ can serve as the values of the probability function of a random variable with the countably infinite range $x = 1, 2, 3, ...$

6. For each of the following, determine whether the given values can serve as the values of a cumulative distribution function of a random variable with the range $x = 1, 2, 3$, and 4:

   (a) $F(1) = 0.3$, $F(2) = 0.5$, $F(3) = 0.8$, and $F(4) = 1.2$;
   (b) $F(1) = 0.5$, $F(2) = 0.4$, $F(3) = 0.7$, and $F(4) = 1.0$;
   (c) $F(1) = 0.25$, $F(2) = 0.61$, $F(3) = 0.83$, and $F(4) = 1.0$.

7. If X has the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1/3 & \text{for } 1 \le x < 4 \\ 1/2 & \text{for } 4 \le x < 6 \\ 5/6 & \text{for } 6 \le x < 10 \\ 1 & \text{for } x \ge 10 \end{cases}$$

   find

(a) $P(2 < X \leq 6)$;
(b) $P(X = 4)$;
(c) the probability function of X.

8. Find the cumulative distribution function of the random variable that has the probability function $f(x) = x/15$ for $x = 1, 2, 3, 4, 5$.

9. Let X be the number of students who show up at a professor's office hours on a particular day. Suppose that the only possible values of X are 0, 1, 2, 3, and 4, and that $f(0) = 0.30$, $f(1) = 0.25$, $f(2) = 0.20$, and $f(3) = 0.15$.

   (a) What is $f(4)$?
   (b) Draw the probability function of X.
   (c) What is the probability that at least two students come to the office hour?

10. A mail-order computer business has six telephone lines. Let X denote the number of lines in use at a specified time. Suppose the probability function of X is as given in the accompanying table.

x	0	1	2	3	4	5	6
$f(x)$	0.10	0.15	0.20	0.25	0.20	0.06	0.04

Calculate the probability of each of the following events.

   (a) {at most three lines are in use}
   (b) {fewer than three lines are in use}
   (c) {at least three lines are in use}
   (d) {between two and five lines, inclusive, are in use}

11. Many manufacturers have quality control programs that include inspection of incoming materials for defects. Suppose a computer manufacturer receives computer boards in lots of five. Two boards are selected from each lot for inspection. We can represent possible outcomes of the selection process by pairs. For example, the pair $(1, 2)$ represents the selection of boards 1 and 2 for inspection.

   (a) List the ten different possible outcomes.
   (b) Suppose that boards 1 and 2 are the only defective boards in a lot of five. Two boards are to be chosen at random. Define X to be the number of defective boards observed among those inspected. Find the probability function of X.
   (c) Obtain the cumulative distribution function of X.

12. A library subscribes to two different weekly news magazines, each of which is supposed to arrive in Wednesday's mail. In actuality, each one may arrive on Wednesday, Thursday, Friday, or Saturday. Suppose the two arrive independently of one another, and for each one $P(W) = 0.3$, $P(Th) = 0.4$, $P(F) = 0.2$, and $P(S) = 0.1$. Let $Y =$ the number of days beyond Wednesday that it takes for both magazines to arrive (so possible Y values are 0, 1, 2, or 3). Compute the probability function of Y. [**Hint:** There are 16 possible outcomes; $Y(W, W) = 0$, $Y(W, Th) = 1$, $Y(F, Th) = 2$, and so on.]

13. The probability density of the continuous random variable X is given by

$$f(x) = \begin{cases} 1/5 & 2 < x < 7 \\ 0 & elsewhere \end{cases}$$

   (a) Draw its graph and verify that the total area under the curve (above the x-axis) is equal to 1.
   (b) Find $P(3 < X < 5)$.

14. Let $f(x) = e^{-x}$ for $0 < x < +\infty$.

   (a) Show that $f(x)$ represents a probability density function.
   (b) Sketch a graph of this function and indicate the area associated with the probability that $X > 1$.
   (c) Calculate the probability that $X > 1$.

15. Let $f(x) = 3x^2$ for $0 < x < 1$.

   (a) Show that $f(x)$ represents a density function.
   (b) Sketch a graph of this function, and indicate the area associated with the probability that $0.1 < X < 0.5$.
   (c) Calculate the probability that $0.1 < X < 0.5$.

16. The probability density function of the random variable X is given by

$$f(x) = \begin{cases} \frac{c}{\sqrt{x}} & 0 < x < 4 \\ 0 & elsewhere \end{cases}$$

   Find

   (a) the value of c;
   (b) $P(X < 14)$ and $P(X > 1)$.

17. The probability density of the random variable Z is given by

$$f(z) = \begin{cases} kze^{-z^2} & z > 0 \\ 0 & z \leq 0 \end{cases}$$

Find k.

18. Find the cumulative distribution function of the random variable X whose probability density is given by

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Also sketch the graphs of the probability density and cumulative distribution functions.

19. Find the cumulative distribution function of the random variable X whose probability density is given by

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \leq 1 \\ 1/2 & \text{for } 1 \leq x \leq 2 \\ (3 - x)/2 & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

Also sketch the graphs of these probability density and cumulative distribution functions.

20. The cumulative distribution function of the random variable Y is given by

$$F(y) = \begin{cases} 1 - \frac{9}{y^2} & \text{for } y > 3 \\ 0 & \text{elsewhere} \end{cases}.$$

Find $P(Y \leq 5)$ and $P(Y > 8)$.

21. The cumulative distribution function of the random variable X is given by

$$F(x) = \begin{cases} 1 - (1 + x)e^{-x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P(X \leq 2)$, $P(1 < X < 3)$, and $P(X > 4)$.

22. The cumulative distribution function of the random variable Z is given by

$$F(z) = \begin{cases} 0 & \text{for } z < -2 \\ \frac{z+4}{8} & \text{for } -2 \leq z < 2 \\ 1 & \text{for } z \geq 2 \end{cases}$$

   (a) Is Z a continuous random variable?

(b) Find $P(Z=-2)$, $P(Z=2)$, $P(-2<Z<1)$, and $P(0\leq Z\leq 2)$.

23. Let X be a random variable with cumulative distribution function

$$F_X(x)=\begin{cases} 0 & \text{for } x\leq 0 \\ \frac{x^2}{2} & \text{for } 0<x\leq 1 \\ 2x-\frac{x^2}{2}-1 & \text{for } 1<x\leq 2 \\ 1 & x>2 \end{cases}$$

(a) What is the cumulative distribution function of $Y=2X+3$?

(b) What is the cumulative distribution function of $Y=-2X+3$?

24. Let $X\sim U(-1,1)$,

$$Y=\begin{cases} 0 & \text{for } X<0 \\ X & \text{for } X\geq 0 \end{cases}, \text{ and } W=\begin{cases} 0 & \text{for } X<0 \\ 1 & \text{for } X\geq 0 \end{cases}.$$

(a) What is the cumulative distribution function of Y?

(b) What is the cumulative distribution function of W?