Stochastic Calculus - Part 10

ISEG

Existence and Uniqueness Theorem for SDE's

- Let $T > 0$, $b(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions such that:

  1) $\mathbb{E}\left[|Z|^2\right] < \infty$ and Z independent of B.

  2) Linear growth property

  $$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|), \; x \in \mathbb{R}^n, \; t \in [0,T]$$

  3) Lipschitz property

  $$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|, \; x,y \in \mathbb{R}^n, \; t \in [0,T]$$

  Then the SDE

  $$X_t = Z + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dB_s \qquad (1)$$

  has a unique solution. Exists a unique stoch. proc. $X = \{X_t, 0 \leq t \leq T\}$ continuous, adapted, which satisfies (1) and

  $$E\left[\int_0^T |X_s|^2\,ds\right] < \infty.$$

Proof of the existence and uniqueness theorem

- Consider the space $L^2_{a,T}$ of processes adapted to the filtration $\mathcal{F}^Z_t := \sigma(Z) \cup \mathcal{F}_t$ such that $E\left[\int_0^T |X_s|^2 \, ds\right] < \infty$.
- In this space, consider the norm:

$$\|X\| = \left(\int_0^T e^{-\lambda s} E\left[|X_s|^2 \right] ds \right)^{\frac{1}{2}},$$

where $\lambda > 2D^2 (T+1)$.
- Define the operator $\mathcal{L} : L^2_{a,T} \to L^2_{a,T}$ by:

$$(\mathcal{L}X)_t = Z + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB_s$$

Proof of the theorem

- By the linear growth of b and σ, the operator $\mathcal{L}$ is well defined.
- By the Cauchy-Schwarz inequality and by Itô isometry, we have:

$$E\left[\left| (\mathcal{L}X)_t - (\mathcal{L}Y)_t \right|^2 \right] \leq 2E\left[\left(\int_0^t (b(s, X_s) - b(s, Y_s)) \, ds \right)^2 \right]$$

$$+ 2E\left[\left(\int_0^t (\sigma(s, X_s) - \sigma(s, Y_s)) \, dB_s \right)^2 \right]$$

$$\leq 2TE\left[\int_0^t (b(s, X_s) - b(s, Y_s))^2 \, ds \right] +$$

$$+ 2E\left[\int_0^t (\sigma(s, X_s) - \sigma(s, Y_s))^2 \, ds \right]$$

Proof of the theorem

- By the Lipschitz property, we have:

$$E\left[\left|(\mathcal{L}X)_t - (\mathcal{L}Y)_t\right|^2\right] \leq 2D^2 (T+1)\, E\left[\int_0^t (X_s - Y_s)^2\, ds\right].$$

- Define $K = 2D^2 (T+1)$. Multiplying the previous inequality by $e^{-\lambda t}$ and integrating in $[0, T]$, we have

$$\int_0^T e^{-\lambda t} E\left[\left|(\mathcal{L}X)_t - (\mathcal{L}Y)_t\right|^2\right] dt$$
$$\leq K \int_0^T e^{-\lambda t} E\left[\int_0^t (X_s - Y_s)^2\, ds\right] dt.$$

  Interchanging the order of integration, we have

$$= K \int_0^T \left[\int_s^T e^{-\lambda t} dt\right] E\left[(X_s - Y_s)^2\right] ds$$
$$\leq \frac{K}{\lambda} \int_0^T e^{-\lambda s} E\left[(X_s - Y_s)^2\right] ds$$

Proof of the theorem

- Therefore

$$\|(\mathcal{L}X) - (\mathcal{L}Y)\| \leq \sqrt{\frac{K}{\lambda}}\, \|X - Y\|$$

- Choosing $\lambda > K$, we have $\sqrt{\frac{K}{\lambda}} < 1$, and the operator $\mathcal{L}$ is a contraction in the space $L^2_{a,T}$. Hence, by the fixed point theorem, exists a unique fixed point to $\mathcal{L}$ and that fixed point is exactly the solution of the SDE:

$$(\mathcal{L}X)_t = X_t.$$

- See the book of Oksendal for a proof based on Picard approximations and the Gronwall inequality.

Examples

- The Geometric Brownian motion

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right]$$

We know that it is the solution of the SDE

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$
$$S_0 = S_0.$$

This SDE models the time evolution of the price of a risky financial asset in the standard Black-Scholes model.

Example

- Consider the Black-Scholes SDE with coefficients $\mu(t)$ and $\sigma(t) > 0$ depending on time:

$$dS_t = S_t\left(\mu(t)\,dt + \sigma(t)\,dB_t\right),$$
$$S_0 = S_0.$$

- How is the solution of this SDE?

Example

- Let $S_t = \exp(Z_t)$ and $Z_t = \ln(S_t)$. By Itô formula with $f(x) = \ln(x)$, we have:

$$\begin{aligned} dZ_t &= \frac{1}{S_t}\left(S_t\left(\mu(t)\,dt + \sigma(t)\,dB_t\right)\right) - \frac{1}{2S_t^2}\left(S_t^2\sigma^2(t)\,dt\right) \\ &= \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)\,dB_t. \end{aligned}$$

  Hence,

$$Z_t = Z_0 + \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)\,dB_s.$$

- Therefore,

$$S_t = S_0 \exp\left(\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)\,dB_s\right).$$

Orsntein-Uhlenbeck process with mean reversion

$$\begin{aligned} dX_t &= a(m - X_t)\,dt + \sigma dB_t, \\ X_0 &= x. \end{aligned}$$

$a, \sigma > 0$ and $m \in \mathbb{R}$.

- Solution of the associated homogeneous ODE $dx_t = -ax_t dt$ is $x_t = xe^{-at}$.
- Consider that the process is such that $X_t = Y_t e^{-at}$ or $Y_t = X_t e^{at}$.
- By the Itô formula applied to $f(t, x) = xe^{at}$, we have

$$Y_t = x + m\left(e^{at} - 1\right) + \sigma \int_0^t e^{as} dB_s.$$

Orsntein-Uhlenbeck process with mean reversion

- Hence,
$$X_t = m + (x - m) e^{-at} + \sigma e^{-at} \int_0^t e^{as} dB_s.$$
- This is a Gaussian process, since it is a stochastic integral of the type $\int_0^t f(s) \, dB_s$, where f is a deterministic function.
- Mean:
$$E[X_t] = m + (x - m) e^{-at}$$

Ornstein-Uhlenbeck process with mean reversion:

- Covariance: by Itô isometry
$$\begin{aligned} \text{Cov}[X_t, X_s] &= \sigma^2 e^{-a(t+s)} E\left[\left(\int_0^t e^{ar} dB_r\right)\left(\int_0^s e^{ar} dB_r\right)\right] \\ &= \sigma^2 e^{-a(t+s)} \int_0^{t \wedge s} e^{2ar} dr \\ &= \frac{\sigma^2}{2a}\left(e^{-a|t-s|} - e^{-a(t+s)}\right). \end{aligned}$$

Note that
$$X_t \sim N\left[m + (x - m) e^{-at}, \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)\right].$$

Ornstein-Uhlenbeck with mean reversion:

- When $t \to \infty$, the distribution of X_t converges to

$$\nu := N\left[m, \frac{\sigma^2}{2a}\right].$$

  which is the invariant or stationary distribution.
- Note that if X_0 has distribution ν then X_t has the same distribution ν for all t.

Financial applications of the Ornstein-Uhlenbeck process with mean reversion

- Vasicek model for the interest rate

$$dr_t = a\left(b - r_t\right)dt + \sigma dB_t,$$

  with a, b, σ parameters.
- Solution:

$$r_t = b + \left(r_0 - b\right)e^{-at} + \sigma e^{-at}\int_0^t e^{as}dB_s.$$

Financial applications of the Ornstein-Uhlenbeck process with mean reversion:

- Black-Scholes model with stochastic volatlity: consider that the volatility $\sigma(t) = f(Y_t)$ is a function of a Ornstein-Uhlenbeck process with mean reversion.

$$dY_t = a(m - Y_t)\,dt + \beta dW_t,$$

  with a, m, β parameters and where $\{W_t, 0 \leq t \leq T\}$ is a Brownian motion.
- The SDE that models the time evolution of the price of the risky asset is

$$dS_t = \mu S_t dt + f(Y_t)\,S_t dB_t$$

  where $\{B_t, 0 \leq t \leq T\}$ is a Brownian motion.and the Brownian motions W_t and B_t may be correlated, i.e.,

$$E[B_t W_s] = \rho(s \wedge t).$$

Example

- Consider the SDE

$$X_t = x + \int_0^t f(s, X_s)\,ds + \int_0^t c(s)\,X_s dB_s,$$

  where f and c are continuous deterministic functions and f satisfies the Lipschitz and linear growth conditions in x.
- By the existence and uniqueness theorem for SDE's, exists one unique solution for this SDE.
- How can we obtain the solution?

Example

- Consider the "integrating factor"

$$F_t = \exp\left(\int_0^t c\,(s)\,dB_s - \frac{1}{2}\int_0^t c\,(s)^2\,ds\right).$$

  Note that F_t is a solution of the SDE if $f = 0$ and $x = 1$.
- Suppose that $X_t = F_t Y_t$ or that $Y_t = (F_t)^{-1} X_t$. Then, by Itô formula,

$$dY_t = (F_t)^{-1} f\,(t, F_t Y_t)\,dt$$

  and $Y_0 = x$.
- This equation for Y is a ODE with random coefficients (is a deterministic ODE parametrized by $\omega \in \Omega$).

Example

- For example, if $f(t,x) = f(t)x$, then we have the ODE

$$\frac{dY_t}{dt} = f(t)Y_t$$

  and therefore

$$Y_t = x\exp\left(\int_0^t f\,(s)\,ds\right).$$

  Hence

$$X_t = x\exp\left(\int_0^t f\,(s)\,ds + \int_0^t c\,(s)\,dB_s - \frac{1}{2}\int_0^t c\,(s)^2\,ds\right).$$

Linear SDE's

- In general, a linear SDE has the form:

$$dX_t = (a(t) + b(t) X_t) \, dt + (c(t) + d(t) X_t) \, dB_t,$$
$$X_0 = x,$$

where a, b, c, d are deterministic continuous functions.

- How to obtain the solution of the SDE?

Linear SDE's

- Assume that

$$X_t = U_t V_t, \tag{2}$$

where

$$\begin{cases} dU_t = b(t) U_t dt + d(t) U_t dB_t, \\ dV_t = \alpha(t) \, dt + \beta(t) \, dB_t. \end{cases}$$

and $U_0 = 1, \; V_0 = x$.

- From a previous example, we know that

$$U_t = \exp\left(\int_0^t b(s) \, ds + \int_0^t d(s) \, dB_s - \frac{1}{2} \int_0^t d(s)^2 \, ds \right) \tag{3}$$

Linear SDE's

- On the other hand, calculating the differential of (2), by Ito's formula with $f(u,v) = uv$, we have

$$dX_t = V_t dU_t + U_t dV_t + \frac{1}{2}(dU_t)(dV_t) + \frac{1}{2}(dV_t)(dU_t)$$
$$= (b(t)X_t + \alpha(t)U_t + \beta(t)d(t)U_t)\,dt + (d(t)X_t + \beta(t)U_t)\,dB_t.$$

- Comparing with the initial SDE for X, we have that

$$a(t) = \alpha(t)U_t + \beta(t)d(t)U_t,$$
$$c(t) = \beta(t)U_t.$$

Linear SDE's

- Hence

$$\beta(t) = c(t)U_t^{-1},$$
$$\alpha(t) = [a(t) - c(t)d(t)]\,U_t^{-1}.$$

- Therefore,

$$X_t = U_t\left(x + \int_0^t [a(s) - c(s)d(s)]\,U_s^{-1}ds + \int_0^t c(s)\,U_s^{-1}dB_s\right),$$

where U_t is given by (3).

SDE's - Theorem of existence and uniqueness for the one-dimensional case

- In the one-dimensional case ($n = 1$), the Lipschitz condition for σ in the existence and uniqueness theorem can be weakened if $\sigma(t, x) = \sigma(x), b(t, x) = b(x)$ (coefficients do not depend on time).
- Assume that b satisfies the Lipschitz condition and the coefficient σ satisfies the condition
$$|\sigma(x) - \sigma(y)| \leq D|x - y|^{\alpha}, \ x, y \in \mathbb{R},$$
with $\alpha \geq \frac{1}{2}$. Then, exists one unique solution for the SDE.
- As an example, the SDE for the Cox-Ingersoll-Ross (CIR) model for interest rates
$$dr_t = a(b - r_t)\,dt + \sigma\sqrt{r_t}dB_t$$
$$r_0 = x,$$
has one and only one solution.

Exercise

- The Cox-Ingersoll-Ross (CIR) model for the interest rate $R(t)$ is given by
$$dR(t) = (\alpha - \beta R(t))\,dt + \sigma\sqrt{R(t)}\,dW(t),$$
where α, β and σ are positive constants. The CIR equation does not have a solution in closed form. However, one can find the mean and the variance of $R(t)$.

a) Calculate the mean value of $R(t)$. (Hint: Let $X(t) = e^{\beta t}R(t)$ and apply the It formula).
b) Calculate the variance of $R(t)$. (Hint: Calculate $d\left(X^2(t)\right)$ using the Itô formula in the differential form and integrate).
c) Calculate $\lim_{t\to+\infty} Var(R(t))$.