ISEG - Lisbon School of Economics and Management
2017/2018

Statistics I
Problem set 4 - Expected values and parameters
(version: 19/04/2018)

1. Find the expected value, the median and the mode of the discrete random variable X having the probability distribution $f(x) = |x-2|/7$, $x = -1, 0, 1, 3$.

2. Find the expected value, the median and the mode, of the random variable Y whose probability density is given by

$$f(y) = \begin{cases} (y+1)/8 & \text{for } 2 \leq y \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

3. Let X be a random variable that has probability density function

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}.$$

   (a) Find the expected value,the median and the mode of the random variable X.

   (b) Compute the variance of $g(X) = 2X + 3$.

4. Let X be a random variable that takes on the values 0, 1, 2, and 3 with probabilities $\frac{1}{125}, \frac{12}{125}, \frac{48}{125}, \frac{64}{125}$.

   (a) Find $E(X)$ and $E(X^2)$

   (b) Use the results of part *(a)* to determine the value of $E[(3X+2)^2]$

5. Let X be a random variable that has probability density function

$$f(x) = \begin{cases} \frac{1}{x\log(3)} & \text{for } 1 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}.$$

   (a) Find $E(X)$, the median and the mode of X.

   (b) Find $E(X^2)$ and $E(X^3)$.

   (c) Use the results of part (a) and (b) to determine $E(X^3 + 2X^2 - 3X + 1)$.

6. Let X be a random variable that has probability density function

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \leq 1 \\ 1/2 & \text{for } 1 < x \leq 2 \\ (3-x)/2 & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find $E(X)$, the median and the mode of X.

(b) Find $E\left(X^2\right)$.

(c) Use the results of part (a) and (b) to determine $E\left(X^2 - 5X + 3\right)$.

7. If the probability distribution of X is given by

$$f(x) = \left(\frac{1}{2}\right)^x, x = 1, 2, 3, ...$$

show that $E(2^X)$ does not exist, but $E(X)$ exists [**Hint:** for any $|c| < 1$, $\sum_{i=1}^{\infty} ic^i = c(1-c)^{-2}$.]

8. Find μ_X, μ_2' and σ_X^2 for the random variable X that has the probability function $f(x) = 1/2$ for for $x = -2$ and $x = 2$.

9. Find μ_X, μ_2' and σ_X^2 for the random variable X that has probability density function

$$f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}.$$

10. If the random variable X has the mean μ and the standard deviation σ, show that the random variable Z whose values are related to those of X by means of the equation

$$Z = \frac{X - \mu}{\sigma}$$

has $E(Z) = 0$ and $var(Z) = 1$. A distribution that has the mean 0 and the variance 1 is said to be in standard form, and when we perform the above change of variable, we are said to be standardizing the distribution of X.

11. If the probability density of X is given by

$$f(x) = \begin{cases} x^{-2} & \text{for } x \geq 1 \\ 0 & \text{elsewhere} \end{cases}.$$

check whether its mean exist. Compute the median and the mode of X.

12. If the probability density of X is given by

$$f(x) = \begin{cases} 2x^{-3} & \text{for } x \geq 1 \\ 0 & \text{elsewhere} \end{cases}.$$

check whether its mean and its variance exist.

13. Consider the following probability functions (which have equal means and standard deviations):

- $f(1) = 0.05$,$f(2) = 0.15$, $f(3) = 0.30$, $f(4) = 0.30$, $f(5) = 0.15$, and $f(6) = 0.05$;

- $f(1) = 0.05$, $f(2) = 0.20$, $f(3) = 0.15$, $f(4) = 0.45$, $f(5) = 0.10$, and$f(6) = 0.05$.

(a) Use the fact that $\mu_3 = \mu_3' - 3\mu_2'\mu_X + 2\mu_X^3$. to compute the parameters of skewness of the probability functions above.

(b) Also draw histograms of the two distributions and note that whereas the first is symmetrical, the second has a "tail" on the left-hand side and is said to be negatively skewed.

14. Use the fact that $\mu_4 = \mu_4' - 4\mu_3'\mu_X + 6\mu_2'\mu_X^2 - 3\mu_X^4$ to compute the parameters of kurtosis for the following following symmetrical distributions, of which the first is more peaked (narrow humped) than the second

    - $f(-3) = 0.06$, $f(-2) = 0.09$, $f(-1) = 0.10$, $f(0) = 0.50$, $f(1) = 0.10$, $f(2) = 0.09$, and $f(3) = 0.06$;
    - $f(-3) = 0.04$, $f(-2) = 0.11$, $f(-1) = 0.20$, $f(0) = 0.30$, $f(1) = 0.20$, $f(2) = 0.11$, and $f(3) = 0.04$.

15. Let X be a continuous random variable with probability density function given by:

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 1/2 & \text{for } 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Compute $E(X)$ and $Var(X)$.

(b) Let $Y = g(X) = 1 - 3X$ and compute $E(Y)$ and $Var(Y)$.

(c) Determine the 1st and 3rd quartiles.

(d) Compute the mean of the following functions of X: $Z = 1/X$ and

$$U = \begin{cases} -1 & \text{for } X < 0.5 \\ 1 & \text{for } X \geq 0.5 \end{cases}$$

16. Let X be a continuous random variable with probability density function given by:

$$f(x) = \begin{cases} x^2/42 & \text{for } -1 < x < 5 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Compute the coefficient of variation of the random variable X.

(b) Determine the median and the interquartile range.

(c) Using the properties of the expected value and of the variance, compute the mean and variance of the random variable $Y = 5 - 3X$.

17. The loss amount, X, for a medical insurance policy has cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{9}\left(2x^2 - \frac{x^3}{3}\right) & \text{for } 0 \leq x < 3 \\ 1 & \text{for } x \geq 3 \end{cases}$$

Calculate the mode of X.

18. Let $X \sim U(0,1)$. Find the moment generating function of X.

19. Find the moment-generating function of the discrete random variable X that has the probability distribution given by

$$f(x) = 2\left(\frac{1}{3}\right)^x, \; x = 1, 2, ...$$

Use it to find the values of μ_1' and μ_2'.

20. If we let $R_X(t) = \log[M_X(t)]$, show that $\left.\frac{\partial R_X(t)}{\partial t}\right|_{t=0} = \mu_X$ and $\left.\frac{\partial^2 R_X(t)}{\partial t^2}\right|_{t=0} = \sigma_X^2$. Also, use these results to find the mean and the variance of a random variable X having the moment generating function $M_X(t) = e^{4(e^t-1)}$.

21. Explain why there can be no random variable for which $M_X(t) = t/(1-t)$.

22. Derive the moment generating function of the random variable has the probability density function $f(x) = e^{-|x|}/2$ for $x \in \mathbb{R}$ and use it to find σ_X^2.

23. Given the moment-generating function $M_X(t) = e^{3t+8t^2}$, find the moment-generating function of the random variable $Z = (X-3)/4$, and use it to determine the mean and the variance of Z.