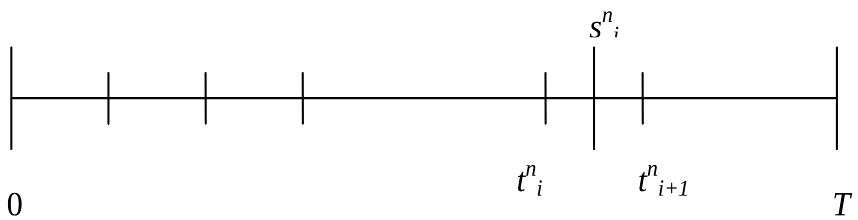
Stochastic Calculus - part 11

ISEG

SDE's - Numerical Approximations

- Many SDE's cannot be solved explicitly $\Longrightarrow$ We need numerical methods to obtain numerical approximations of the solutions
- SDE:
$$dX_t = b\left(X_t\right)dt + \sigma\left(X_t\right)dB_t,$$
with initial condition $X_0 = x$.
- Partition: $t_i = \frac{iT}{n}$, $i = 0, 1, \ldots, n$ and length of each sub-interval: $\delta_n = \frac{T}{n}$



Euler Method

- Exact values of the solution:

$$X(t_i) = X(t_{i-1}) + \int_{t_{i-1}}^{t_i} b(X_s)\, ds + \int_{t_{i-1}}^{t_i} \sigma(X_s)\, dB_s. \tag{1}$$

- Euler approximation

$$\int_{t_{i-1}}^{t_i} b(X_s)\, ds \approx b(X(t_{i-1}))\, \delta_n,$$
$$\int_{t_{i-1}}^{t_i} \sigma(X_s)\, dB_s \approx \sigma(X(t_{i-1}))\, \Delta B_i,$$

where $\Delta B_i := B(t_i) - B(t_{i-1})$.

- Euler scheme:

$$X^{(n)}(t_i) = X^{(n)}(t_{i-1}) + b\left(X^{(n)}(t_{i-1})\right)\delta_n + \sigma(X(t_{i-1}))\, \Delta B_i, \tag{2}$$

$i = 1, 2, \ldots, n.$

Euler Method

- In each interval (t_{i-1}, t_i), the value of $X^{(n)}$ is obtained by linear interpolation.
- The approximation error is defined by

$$e_n := \sqrt{E\left[\left(X_T - X_T^{(n)}\right)^2\right]}. \tag{3}$$

- For the Euler scheme, we can show that

$$e_n^{Eul} \le c\sqrt{\delta_n},$$

where c is a constant.

Euler Method

- How to simulate a trajectory of the solution using the Euler method?

1. We have to generate the values of n random variables with normal distribution $N(0,1)$: $\xi_1, \xi_2, \ldots, \xi_n$.
2. Replace ΔB_i in (2) by $\xi_i \sqrt{\delta_n}$ and calculate the values of $X^{(n)}(t_i)$ using the recurrence scheme (2).
3. In each interval (t_{i-1}, t_i) calculate $X^{(n)}$ by linear interpolation between $X^{(n)}(t_{i-1})$ and $X^{(n)}(t_i)$.

Milstein Method

- Apply the Itô formula to $b(X_t)$ and $\sigma(X_t)$, considering $t_{i-1} \leq t \leq t_i$. We have

$$\int_{t_{i-1}}^{t_i} b(X_s)\, ds = \int_{t_{i-1}}^{t_i} \left[b(X(t_{i-1})) + \right.$$

$$\left. + \int_{t_{i-1}}^{s} \left(bb' + \frac{1}{2} b'' \sigma^2 \right)(X_r)\, dr + \int_{t_{i-1}}^{s} (\sigma b')(X_r)\, dB_r \right] ds,$$

$$\int_{t_{i-1}}^{t_i} \sigma(X_s)\, dB_s = \int_{t_{i-1}}^{t_i} \left[\sigma(X(t_{i-1})) + \right.$$

$$\left. + \int_{t_{i-1}}^{s} \left(b\sigma' + \frac{1}{2} \sigma'' \sigma^2 \right)(X_r)\, dr + \int_{t_{i-1}}^{s} (\sigma\sigma')(X_r)\, dB_r \right] dB_s.$$

- Exercise: Prove this equality.

Milstein Method

- Then, from (1), we have that

$$X^{(n)}(t_i) - X^{(n)}(t_{i-1}) = b\left(X\left(t_{i-1}\right)\right)\delta_n + \sigma\left(X\left(t_{i-1}\right)\right)\Delta B_i + R_i.$$

One can show that the dominant term of R_i is the double stochastic integral

$$\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\left(\sigma\sigma'\right)\left(X_r\right)dB_r\right)dB_s,$$

while the other terms are of higher order and negligible.

- Milstein Approximation:

$$\begin{aligned} R_i &\approx \int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\left(\sigma\sigma'\right)\left(X_r\right)dB_r\right)dB_s \\ &\approx \left(\sigma\sigma'\right)\left(X\left(t_{i-1}\right)\right)\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}dB_r\right)dB_s \end{aligned}$$

and

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$$\begin{aligned} &\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}dB_r\right)dB_s = \int_{t_{i-1}}^{t_i}\left(B_s - B\left(t_{i-1}\right)\right)dB_s \\ &= \int_{t_{i-1}}^{t_i}B_s dB_s - B\left(t_{i-1}\right)\left(B\left(t_i\right) - B\left(t_{i-1}\right)\right) \\ &= \frac{1}{2}\left[B_{t_i}^2 - B_{t_{i-1}}^2 - \delta_n\right] - B\left(t_{i-1}\right)\left(B\left(t_i\right) - B\left(t_{i-1}\right)\right) \\ &= \frac{1}{2}\left[\left(\Delta B_i\right)^2 - \delta_n\right], \end{aligned}$$

where, for calculating $\int_{t_{i-1}}^{t_i} B_s dB_s$, we can use the Itô formula applied to $f(B_t) = B_t^2$.

Milstein Method

- Milstein Scheme:

$$X^{(n)}(t_i) = X^{(n)}(t_{i-1}) + b\left(X^{(n)}(t_{i-1})\right)\delta_n + \sigma(X(t_{i-1}))\Delta B_i$$
$$+ \frac{1}{2}(\sigma\sigma')(X(t_{i-1}))\left[(\Delta B_i)^2 - \delta_n\right].$$

- One can show that the approximation error in the Milstein methos is

$$e_n^{Mil} \le c\delta_n.$$