ISEG - Lisbon School of Economics and Management
2017/2018

Statistics I
Problem set 5 - Expected values and parameters of functions of random variables
(version: 2/05/2018)

1. If X and Y have the joint probability distribution $f(x,y) = 1/4$ for $x = -3$ and $y = -5$, $x = -1$ and $y = -1$, $x = 1$ and $y = 1$, and $x = 3$ and $y = 5$, find $cov(X,Y)$.

2. Let X and Y be continuous and independent random variables. Prove that $E\left[h(X)\,v(Y)\right] = E\left(h(X)\right)E\left(v(Y)\right)$ for any two functions $h(X)$ and $v(Y)$, provided that $E\left(|h(X)|\right)$ and $E\left(|v(Y)|\right)$ are finite.

3. If X and Y have the joint probability distribution $f(-1,0) = 0$, $f(-1,1) = 1/4$, $f(0,0) = 1/6$, , $f(0,1) = 0$, $f(1,0) = 1/12$, and $f(1,1) = 1/2$, show that

   (a) $cov(X,Y) = 0$;

   (b) the two random variables are not independent.

4. Let $Y = bX + a$, where b and a are constants, and $\rho_{X,Y}$ be the correlation coefficient between X and Y. Prove that if $Var(X) > 0$:

   (a) $\rho_{X,Y} = 1$ if $b > 0$.

   (b) $\rho_{X,Y} = -1$ if $b < 0$.

   (c) If $b = 0$, it is not defined.

5. In a certain shop which sells computer components, the daily sales of hard drives of brands X and Y has the following joint probability function:

$y \setminus x$	0	1	2
0	0.12	0.25	0.13
1	0.05	0.30	0.01
2	0.03	0.10	0.01

   (a) Compute the means and variances of X and Y.

   (b) Analyze the independence of the two random variables and compute the correlation coefficient.

   (c) Find that $E(Y|X = x)$ is not equal to $E(Y)$. Comment.

   (d) Compute the mean and variance of $Z = X - Y$.

6. Let (X,Y) be a discrete random vector with joint probability function given by:

$$f(x,y) = \frac{x+y}{32},\ \ x = 1,2;\ y = 1,2,3,4$$

(a) Compute the means and variances of X and Y

(b) Using $E(XY)$ analyze the independence of the two random variables and compute the correlation coefficient.

(c) Compute $E(X|Y = y)$

7. Let (X, Y) be a discrete random vector with joint probability function:

$y \setminus x$	-1	0	1
-1	b	0	c
0	0	a	0
1	c	0	b

(a) Find a, b and c such that X and Y are not correlated.

(b) Find a, b and c such that there is a perfect correlation between X and Y

(c) Compute the mean and variance of $Z = |X - Y|$.

8. Let and X and Y be independent random variables with variances σ_X^2 and σ_Y^2. Let $Z = X + Y$ and $W = X - Y$. Derive the expression of $\rho_{Z,W}$.

9. A company sells items A and B. The monthly sales of of items A and B, expressed in monetary units, constitute a random vector , with joint probability density function given by:

$$f(x, y) = \begin{cases} \frac{1}{2} & \text{, for } 0 < x \leq 2,\ 0 < y < x \\ 0 & \text{, elsewhere} \end{cases}$$

(a) Compute the means and variances of X and of Y.

(b) Analyze the independence of the two random variables and compute the correlation coefficient.

(c) Find the $E(Y|X = 1)$

(d) Compute the mean and variance of total sales of the two items.

10. Consider the random vector (X, Y) with joint probability density function defined by:

$$f(x, y) = \begin{cases} 8xy & \text{, for } 0 < x \leq 1,\ 0 < y < x \\ 0 & \text{, elsewhere} \end{cases}$$

(a) Compute the means and variances of X and of Y.

(b) Analyze the independence of the two random variables and compute the correlation coefficient.

(c) Find the $E(X|Y = y)$.

11. If the probability density of X is given by

$$f(x) = \begin{cases} 1+x & ,\text{for } -1 < x \leq 0 \\ 1-x & ,\text{for } 0 < x < 1 \\ 0 & ,\text{elsewhere} \end{cases}$$

and $U = X$ and $V = X^2$, show that

(a) $cov(U, V) = 0$;

(b) U and V are dependent.

12. Let X_1,X_2, and X_3 be independent random variables with means 4, 9, and 3 and the variances 3, 7, and 5.

(a) Find the means and the variances of $Y = 2X_1 - 3X_2 + 4X_3$ and $Z = X_1 + 2X_2 - X_3$.

(b) Find $cov(Y, Z)$.

13. Repeat both parts of exercise 10, dropping the assumption of independence and using instead the information that $cov(X_1, X_2) = 1$, $cov(X_2, X_3) = -2$, and $cov(X_1, X_3) = -3$.

14. If the joint probability density of X and Y is given by

$$f(x, y) = \begin{cases} \frac{1}{3}(y+x) & ,\text{for } 0 < x \leq 1,\ 0 < y < 2 \\ 0 & ,\text{elsewhere} \end{cases}$$

find the variance of $W = 3X + 4Y - 5$.

15. Suppose that only two factories supply light bulbs to the market. Factory A's bulbs work for an average of 5000 hours, whereas factory B 's bulbs work for an average of 4000 hours. It is known that factory A supplies 60% of the total bulbs available. What is the expected length of time that a purchased bulb will work for?

16. Suppose that $E(Y|X = x) = 0.7 + 0.002x$. If the $E(X)$ is 1100 and $Var(X) = 290\,000$, what is $cov(Y, X)$?

17. (*The mean of the sum of a random number of random variables*) Let $W = \sum_{i=1}^{N} X_i$, where both $X_i, i = 1, 2, ...$ and N are random variables. Suppose that the $X_i's$ are independent of N and that $E(X_i) = \mu_X$ for all i, and that $E(N) = \mu_N$. Prove that $E(W) = \mu_X \times \mu_N$.