Stochastic Calculus - Part 15

ISEG

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Kolmogorov Equations

- The Kolmogorov equations are partial differential equations for the transition probabilities of the solution of a stochastic differential equation (diffusion).
- The forward Kolmogorov equation is also known as the Fokker–Planck equation. In natural sciences, the forward equation is also known as "master equation".

- Assume that the process X is a solution of the stochastic differential equation
$$dX_t = b(t, X_t)dt + \sigma(t, X_t)\, dB_t, \tag{1}$$
with associated infinitesimal generator
$$Af(s,y) = \sum_{i=1}^{n} b_i(s,y) \frac{\partial f}{\partial y_i}(s,y) \tag{2}$$
$$+ \frac{1}{2} \sum_{i.j=1}^{n} \left[\sigma(s,y)\sigma^T(s,y)\right]_{i,j} \frac{\partial^2 f}{\partial y_i \partial y_j}(s,y), \tag{3}$$
or, in the one-dimensional case:
$$Af(s,y) = b(s,y)\frac{\partial f}{\partial y}(s,y) + \frac{1}{2}\sigma^2(s,y)\frac{\partial^2 f}{\partial y^2}(s,y) \tag{4}$$

Kolmogorov Equations

- Consider the Boundary value problem (PDE+boundary conditions):
$$\left(\frac{\partial u}{\partial s} + Au\right)(s,y) = 0 \quad \text{if } (s,y) \in]0,T[\times \mathbb{R}^n, \tag{5}$$
$$u(T,y) = \mathbf{1}_C(y) \quad \text{if } y \in \mathbb{R}^n.$$
By the Feynman-Kac formula, we know that
$$u(s,y) = \mathbb{E}_{s,y}\left[\mathbf{1}_C(X_T)\right] = \mathbb{P}\left[X_T \in C | X_s = y\right] = P(C,T,y,s),$$
where
$$\begin{cases} dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dB_t \\ X_s = y \end{cases}$$
and $P(C,T,y,s)$ are the transition probabilities associated to the Markov process X from time s to time T.

Kolmogorov Backward Equation

Theorem

(Kolmogorov Backward Equation) Let X be a solution of (1). Then, the transition probabilities $P(C,t,y,s) = \mathbb{P}[X_t \in C | X_s = y]$ are solutions of

$$\begin{cases} \left(\frac{\partial P}{\partial s} + AP\right)(C,t,s,y) = 0 \quad \text{if } (s,y) \in]0,t[\times \mathbb{R}^n, \\ P(C,t,y,t) = \mathbf{1}_C(y) \quad \text{if } y \in \mathbb{R}^n. \end{cases} \tag{6}$$

- If the transition measure $P(dx,t,y,s)$ has a probability density function $\widetilde{f}(x,t,y,s)\,dx$, then $\widetilde{f}(x,t,y,s)$ is a solution of

$$\begin{cases} \left(\frac{\partial \widetilde{f}}{\partial s} + A\widetilde{f}\right)(x,t,s,y) = 0 \quad \text{if } (s,y) \in]0,t[\times \mathbb{R}^n, \\ \widetilde{f}(x,t,y,s) \longrightarrow \delta_x \ \text{ when } s \nearrow t. \end{cases} \tag{7}$$

- These equations are called "backward" because the differential operator A applies to the "backward" variables (s,y) and not to the forward variables (x,t).

Kolmogorov Forward Equation

- Consider the one dimensional case in order to have a simple notation. Let $s < T$ and let $h(t,x) \in C_c^{\infty}(]s,T[\times \mathbb{R})$ be a smooth function (test function) of compact support in $]s,T[\times \mathbb{R}$.
- By the Itô formula, we have

$$h(T,X_T) = h(s,X_s) + \int_s^T \left(\frac{\partial h}{\partial t} + Ah\right)(t,X_t)\,dt + \int_s^T \frac{\partial h}{\partial x}(t,X_t)\,dB_t.$$

Apllying the conditional expectation $\mathbb{E}_{s,y}[\cdot] = \mathbb{E}[\cdot | X_s = y]$, and using the fact that $h(T,x) = h(s,x) = 0$ (because $h(t,x)$ has compact support in $]s,T[\times \mathbb{R}$) and the zero mean property of the stochastic integral, we obtain

$$\int_{-\infty}^{+\infty}\int_s^T \left(\frac{\partial}{\partial t} + b(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}\right) \times h(t,x)\,\widetilde{f}(x,t,y,s)\,dtdx = 0.$$

Kolmogorov Forward Equation

- If we integrate by parts with respect to t (for the $\frac{\partial}{\partial t}$ part) and by parts with respect to x (for the $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ parts), we obtain:

$$\int_{-\infty}^{+\infty}\int_{s}^{T} h(t,x)\left(-\frac{\partial}{\partial t}\widetilde{f}(x,t,y,s)-\frac{\partial}{\partial x}\left[b(t,x)\widetilde{f}(x,t,y,s)\right]\right.$$
$$\left.+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(t,x)\widetilde{f}(x,t,y,s)\right]\right)dtdx=0.$$

- This equation must hold for all test functions $h(t,x)\in C_c^{\infty}(]s,T[\times\mathbb{R})$, and therefore:

$$-\frac{\partial}{\partial t}\widetilde{f}(x,t,y,s)-\frac{\partial}{\partial x}\left[b(t,x)\widetilde{f}(x,t,y,s)\right]$$
$$+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(t,x)\widetilde{f}(x,t,y,s)\right]=0.$$

Kolmogorov Forward Equation

Theorem

(Kolmogorov Forward Equation): Let X be a solution of (1) with transition probability density function $\widetilde{f}(x,t,y,s)$. Then $\widetilde{f}$ satisfies the equation

$$\begin{cases}\frac{\partial\widetilde{f}}{\partial t}(x,t,s,y)=A^*\widetilde{f}(x,t,y,s) \text{ if } (t,x)\in\,]s,T[\times\mathbb{R},\\ \widetilde{f}(x,t,y,s)\longrightarrow\delta_y \text{ when } t\searrow s,\end{cases}\tag{8}$$

where the operator A^* is the adjoint operator of A and is defined by

$$(A^*f)(t,x)=-\frac{\partial}{\partial x}\left[b(t,x)f(t,x)\right]+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(t,x)f(t,x)\right].\tag{9}$$

- The Kolmogorov forward equation is also known as the Fokker-Planck equation.

Kolmogorov Forward Equation

- In the multidimensional case, the Kolmogorov forward equation is

$$\frac{\partial \widetilde{f}}{\partial t}(x,t,s,y) = A^* \widetilde{f}(x,t,y,s) \text{ if } (t,x) \in \left]s,T\right[\times \mathbb{R}^n,$$

where the adjoint operator A^* is defined by

$$\begin{aligned}(A^* f)(t,x) = &-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[b_i(t,x) f(t,x)\right] \\ &+ \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_i}\left[\left[\left[\sigma(t,x)\sigma^T(t,x)\right]_{i,j}\right] f(t,x)\right].\end{aligned}$$

- Note that in the forward equation, the adjoint operator applies to the "forward" variables (x,t).

Example

- Consider the stochastic differential equation

$$\begin{aligned} dX_t &= \sigma dB_t, \\ X_s &= y, \end{aligned}$$

where σ is a constant. The Fokker-Planck equation for this process is

$$\frac{\partial \widetilde{f}}{\partial t}(x,t,s,y) = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left[\widetilde{f}(x,t,s,y)\right],$$

and the solution is given by the Gaussian probability density function

$$\widetilde{f}(x,t,s,y) = \frac{1}{\sigma\sqrt{2\pi(t-s)}} \exp\left[-\frac{(x-y)}{2\sigma^2(t-s)}\right].$$

Example

- Consider the stochastic differential equation for the geometric Brownian motion

$$dX_t = \alpha X_t dt + \sigma X_t dB_t,$$
$$X_s = y.$$

The Fokker-Planck equation for this process is

$$\frac{\partial \widetilde{f}}{\partial t}(x,t,s,y) = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left[x^2 \widetilde{f}(x,t,s,y)\right] - \alpha \frac{\partial}{\partial x}\left[x\widetilde{f}(t,x)\right],$$

or

$$\frac{\partial \widetilde{f}}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \widetilde{f}}{\partial x^2} + \left(2\sigma^2 - \alpha\right) x \frac{\partial \widetilde{f}}{\partial x} + \left(\sigma^2 - \alpha\right)\widetilde{f}.$$