PhD in Mathematics Applied to Economics and Management
(Course: Mathematical methods for finance)

Stochastic Calculus

Final Exam; Exam duration: 2 hours; June 2, 2014

Justify your answers

1. Consider the Brownian motion $B = \{B_t, t \geq 0\}$.
(a) The process

$$X_t = B_t^3 - 3t + B_t^6 - 15 \int_0^t B_s^2 ds,$$

is it a martingale? Justify your answer.
(b) Consider the process Y defined by

$$Y_t = B_{3t} - B_{2t}.$$

Calculate the mean and variance of Y_t. Is Y a Gaussian process? And is it a Brownian motion? Justify your answers.

2. Let u_t e v_t be two processes adapted to the filtration generated by the Brownian motion $B = \{B_t, t \in [0,T]\}$, such that $u, v \in L^2_{a,T}$. Show that

$$\operatorname{cov}\left[\int_0^t u_s dB_s, \int_0^t v_s dB_s\right] = \int_0^t \mathbb{E}\left[u_s v_s\right] ds$$

and calculate

$$\operatorname{cov}\left[\int_0^T \left(B_t\left(e^t + 1\right)\right) dB_t, \int_0^T \left(B_t e^{-t}\right) dB_t\right].$$

(Hint: you can apply the formula: $xy = \frac{1}{2}\left(x+y\right)^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$)

3. Let c and σ be real constants and $B = \{B_t, t \in [0,T]\}$ a Brownian motion. Let $f(t,x)$ be the solution of the ordinary differential equation (O.D.E.)

$$\begin{cases} \frac{du}{dt} = c + u\left(t\right), \\ u\left(0\right) = x. \end{cases}$$

(a) Solve the equation, determine $f(t,x)$ and find the process $X = \{X_t, t \in [0,T]\}$, in such a way that the process $Y_t = f(t, X_t)$ is a solution of the S.D.E.

$$\begin{aligned} dY_t &= (c + Y_t)dt + \sigma dB_t, \\ Y_0 &= x. \end{aligned}$$

(b) Calculate the mean and the variance of process Y.

4. Consider the boundary value problem with domain $[0,T] \times \mathbb{R}$:

$$\begin{aligned} \frac{\partial F}{\partial t} + 2x\frac{\partial F}{\partial x} &= 3F\left(t,x\right) - 18\frac{\partial^2 F}{\partial x^2}, \quad t > 0,\ x \in \mathbb{R} \\ F\left(T,x\right) &= 1 + x^2. \end{aligned}$$

Specify the infinitesimal generator of the associated diffusion, obtain an explicit expression for this diffusion process, write the stochastic representation formula for the solution of the problem and obtain an expression for the solution of the boundary value problem (as explicit as you can).

5. Consider the Black-Scholes model, with one risky asset S_t and one riskless asset B_t. Assets S_t and B_t have dynamics given by the SDE's

$$dS_t = \mu S_t\, dt + \sigma\, S_t\, d\overline{W}_t \quad \text{and} \quad dB_t = r\, B_t\, dt, \text{ with } B_0 = 1,$$

where $\overline{W}$ is a Brownian motion.

(a) Consider one contingent claim with payoff $\chi = \Phi(S_T)$ and let $G_t = F(t, S_t)$ be the price of this contingent claim (financial derivative) at time t. Show that, under the martingale measure $\mathbb{Q}$, the price G_t satisfies the SDE

$$dG_t = rG_t dt + u_t dW_t,$$

where W_t is a Brownian motion with respect to $\mathbb{Q}$. Obtain also an expression for the process u_t in terms of σ, S_t and $F(t, S_t)$ or its derivatives.

(b) Calculate the price (at time $t < T_1$) of the contingent claim with payoff

$$\chi = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S_t dt,$$

where $T_1 < T_2$. The interval $[T_1, T_2]$ is called the effective period of the financial derivative (T_2 is the maturity date).

Hint: even if the payoff does not have the form $\chi = \Phi(S_T)$, you can assume that the price of this derivative can be obtained using the formula

$$\Pi(t) = e^{-r(T_2 - t)} \mathbb{E}^{\mathbb{Q}}_{t,s}[\chi].$$

6. Let $B = \{B_t, t \in [0, T]\}$ be a Brownian motion. Consider the Itô process

$$X_t = x + \int_0^t v_s ds + \int_0^t u_s dB_s,$$

where v and u are defined in such a way that both integrals exist and are well defined.

Assume that

$$|v_t| \leq M(1 + |X_t|), \quad \forall t \geq 0,$$

where M is a positive constant.

Define the process $X_t^* := \sup_{0 \leq s \leq t} |X_s|$ and prove that

$$X_T^* \leq \left(|x| + MT + \sup_{0 \leq t \leq T} \left| \int_0^t u_s dB_s \right| \right) e^{MT}.$$

Hint: at the end of the proof, you may consider the process $Y_t := \int_0^t X_s^* ds$ or you can apply the Gronwall inequlity.

Marks: 1(a):2.0, (b):2.0, 2:2.5, 3(a):2.5, (b):2.0, 4:2.5, 5(a):2.0, (b):2.5, 6:2.0