Master in Mathematical Finance, ISEG, University of Lisbon

Stochastic Calculus

Final Exam; Exam duration: 2 hours; June 3, 2016

Justify your answers and calculations

1. Consider a standard Brownian motion $B = \{B_t, t \geq 0\}$.

(a) What must be the relationship between the parameters a and b such that the process

$$X_t = aB_t^3 + btB_t$$

is a martingale? Justify your answer.

(b) Consider that we have 2 other standard Brownian motions: $W^{(1)}(t)$, $W^{(2)}(t)$ and that all the Brownian motions are independent. Define the process Y by

$$Y_t = \frac{B_t + W^{(1)}(t) + \frac{1}{3}W^{(2)}(9t)}{\sqrt{3}}$$

Is Y a Gaussian process? And is it a Brownian motion? Justify your answers.

2. Consider a standard Brownian motion $B = \{B_t, t \geq 0\}$. Find $z \in \mathbb{R}$ and a process $u \in L^2_{a,T}$ such that

$$e^{\frac{T}{2}} \cosh(B_T) = z + \int_0^T u_s dB_s.$$

(note that $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and you can try to apply the Itô formula to $e^{B_t - \frac{t}{2}}$ and $e^{-B_t + \frac{t}{2}}$).

3. Let $B^{(1)}$ and $B^{(2)}$ be independent Brownian motions and consider the processes X and Y which are solutions of the stochastic differential equations:

$$\begin{cases} dX_t = \alpha X_t dB_t^{(1)} + \beta X_t dB_t^{(2)}, & X_0 = x_0, \\ dY_t = \gamma Y_t dt + \sigma Y_t dB_t^{(1)}, & Y_0 = y_0, \end{cases}$$

where α, β, γ and σ are constants.

(a) Show that these stochastic differential equations have one unique solution.

(b) Compute $\mathbb{E}[X_t Y_t]$ without solving explicitly the equations.

4. Consider the boundary value problem with domain $[0, T] \times \mathbb{R}$:

$$\frac{\partial F}{\partial t} + 4x\frac{\partial F}{\partial x} = \frac{1}{2}F(t,x) - 32x^2\frac{\partial^2 F}{\partial x^2}, \quad t > 0,\ x \in \mathbb{R}$$
$$F(T,x) = x^2.$$

Specify the infinitesimal generator of the associated diffusion, obtain an explicit expression for this diffusion process, write the stochastic representation formula for the solution of the problem and obtain an expression for the solution of the boundary value problem (as explicit as you can).

5. Consider the Black-Scholes model, with one risky asset S_t and one riskless asset B_t. Assets S_t and B_t have dynamics given by the SDE's

$$dS_t = \mu S_t \, dt + \sigma \, S_t \, d\overline{W}_t \quad \text{and} \quad dB_t = r \, B_t \, dt, \text{ with } B_0 = 1,$$

where $\overline{W}$ is a Brownian motion.

(a) Consider one contingent claim with payoff $\chi = \Phi(S_T)$ and let $G_t = F(t, S_t)$ be the price of this contingent claim (financial derivative) at time t. Show that, under the martingale measure $\mathbb{Q}$, the price G_t satisfies the SDE

$$dG_t = rG_t dt + u_t dW_t,$$

where W_t is a Brownian motion with respect to $\mathbb{Q}$. Obtain also an expression for the process u_t in terms of σ, S_t and $F(t, S_t)$ or its derivatives.

(b) Consider a financial derivative Φ that has the following payoff at expiry date T depending on the price of the underlying non-dividend paying share at maturity T:

$$Payoff = \begin{cases} K \text{ if } S_T > e^K, \\ \ln[S(T)] \text{ if } S_T \leq e^K, \end{cases}$$

where K is positive constant. Show that the price of the derivative at time t is given by (for $t < T$)

$$e^{-r(T-t)} \int_{-\infty}^{K^*} \left(\ln(S_t) + \left(r - \frac{\sigma^2}{2} \right)(T-t) + \sigma z \sqrt{T-t} \right) f(z) dz$$
$$+ K e^{-r(T-t)} \left[1 - \Phi(K^*) \right],$$

where K^* is an appropriate constant, $f(z)$ is the probability density function and $\Phi(z)$ is the cumulative distribution function of the $N(0,1)$ distribution.

6. Let $u \in L^2_{a,T}$ and let $u^{(n)}$ be a sequence of simple processes such that

$$\lim_{n \to \infty} E \left[\int_0^T \left| u_t - u_t^{(n)} \right|^2 dt \right] = 0.$$

Let $M_n(t) := \int_0^t u_s^{(n)} \mathrm{d}B_s$ and prove that

$$\lim_{m,n \to \infty} P \left[\sup_{0 \leq t \leq T} |M_n(t) - M_m(t)| > \lambda \right] = 0$$

for any $\lambda > 0$. Moreover, explain how we can deduce from this result that exists a sequence positive integers n_k, $k = 1, 2, \ldots$, such that

$$\sum_{k=1}^{\infty} P \left[\sup_{0 \leq t \leq T} \left| M_{n_{k+1}}(t) - M_{n_k}(t) \right| > 2^{-k} \right] < \infty.$$

Hint: For the first part, you can use the Doob maximal inequality (or martingale inequality).

Marks: 1(a):2.0, (b):2.0, 2:2.5, 3(a):2.0, (b):2.5, 4:2.5, 5(a):2.0, (b):2.5, 6:2.0