Master in Mathematical Finance, ISEG, University of Lisbon

Stochastic Calculus

Final Exam; Exam duration: 2 hours; July 1, 2016

Justify your answers and calculations

1. Consider a standard Brownian motion $B = \{B_t, t \geq 0\}$.

(a) Consider the process

$$X_t = (B_t + t) \exp\left(-B_t - \frac{1}{2}t\right).$$

Is it a martingale? Justify your answer.

(b) Consider the process Y defined by $Y_t := B_{3t} - B_t$, with $t \geq 0$. Is Y a Gaussian process? And is it a Brownian motion? Justify your answers.

2. Let $u = \{u_t, t \in [0,T]\}$ and $v = \{v_t, t \in [0,T]\}$ be measurable stochastic processes adapted to the filtration generated by the Brownian motion $B = \{B_t, t \in [0,T]\}$, and such that $\mathbb{E}\left[\int_0^T u_t^2 dt\right] < \infty$, $\mathbb{E}\left[\int_0^T v_t^2 dt\right] < \infty$.

Show that

$$\operatorname{cov}\left(\int_0^T (u_t + v_t)\, dB_t, \int_0^T (u_t - v_t)\, dB_t\right) = \int_0^T \left(\mathbb{E}\left[u_t^2\right] - \mathbb{E}\left[v_t^2\right]\right) dt.$$

and calculate

$$\operatorname{cov}\left(\int_0^T \left(B_t e^t + \sqrt{t}\right) dB_t, \int_0^T \left(B_t e^t - \sqrt{t}\right) dB_t\right)$$

(Hint: you can use the identity $ab = \frac{1}{2}(a+b)^2 - \frac{1}{2}a^2 - \frac{1}{2}b^2$).

3. Let X be a stochastic process that satisfies the SDE

$$dX_t = \left(5 + X_t^2\right)^{\frac{1}{2}} dB_t, \qquad 0 \leq t \leq T,$$
$$X_0 = 2.$$

(a) Show that exists one unique solution to this SDE and calculate the mean and the variance of X_t.

(b) Let $h(u,t) := E\left[e^{uX_t}\right]$. Show that $h(u,t)$ satisfies the PDE

$$\frac{\partial h(u,t)}{\partial t} = \frac{1}{2}u^2\left(5h(u,t) + \frac{\partial^2 h(u,t)}{\partial^2 u}\right), \quad \text{and } h(u,0) = e^{2u}.$$

(assume that the undefined stochastic integrals are martingales and that $\mathbb{E}\left[\frac{\partial^2 h(u,t)}{\partial^2 u}\right] = \frac{\partial^2}{\partial^2 u}\mathbb{E}\left[h(u,t)\right]$.)

4. Consider the boundary value problem with domain $[0,T]\times\mathbb{R}$:

$$\frac{\partial F}{\partial t} = 3F(t,x) - 8x\frac{\partial F}{\partial x} - 50x^2\frac{\partial^2 F}{\partial x^2}, \quad t>0,\ x\in\mathbb{R}$$
$$F(T,x) = \ln\left(x^2\right) + x.$$

Specify the infinitesimal generator of the associated diffusion, obtain an explicit expression for this diffusion process, write the stochastic representation formula for the solution of the problem and obtain an expression for the solution of the boundary value problem (as explicit as you can).

5. Consider the Black-Scholes model, with one risky asset S_t and one riskless asset B_t. Assets S_t and B_t have dynamics given by the SDE's

$$dS_t = \mu S_t\,dt + \sigma\,S_t\,d\overline{W}_t \quad \text{and} \quad dB_t = r\,B_t\,dt, \text{ with } B_0 = 1,$$

where $\overline{W}$ is a Brownian motion.

(a) Using the Girsanov Theorem, show that exists one probability measure $\mathbb{Q}$, such that in the probability space $(\Omega,\mathcal{F}_T,\mathbb{Q})$, the process S_t has dynamics (under $\mathbb{Q}$) given by the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where W_t is a standard Brownian motion (under the measure $\mathbb{Q}$). Specify also what is the density L_t of $\mathbb{Q}$ with respect to $\mathbb{P}$.

(b) Consider a financial derivative Φ with payoff

$$\chi = \frac{1}{T}\int_0^T S_u du,$$

where T is the maturity date and S_u is the price of the underlying non-dividend paying share at time u. Assuming that you can use the risk neutral valuation formula

$$\Pi(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}_{t,s}\left[\chi\right],$$

calculate the price of this financial derivative at time $t=0$.

6. Let $B=\{B_t, t\geq 0\}$ be a Brownian motion and b and σ be measurable functions that satisfy the Lipschitz condition

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D\,|x-y|, \ \forall x,y\in\mathbb{R},\ \forall t\in[0,T].$$

Show that if the SDE

$$X_t = \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dB_s \tag{1}$$

has two solutions X_1 and X_2, then

$$P\left[X_1(t) = X_2(t) \text{ for all } t\in[0,T]\right] = 1.$$

Hint: You can use the Gronwall inequality: If $v(t)$ is a nonnegative function such that

$$v(t) \leq C + A\int_0^t v(s)ds$$

for all $t \in [0, T]$, for some constant A and C, then

$$v(t) \le C \exp(At)$$

for all for all $t \in [0, T]$.

Marks: 1(a):2.0, (b):2.0, 2:2.5, 3(a):2.0, (b):2.5, 4:2.5, 5(a):2.0, (b):2.5, 6:2.0