Master in Mathematical Finance, ISEG, University of Lisbon

# Stochastic Calculus

Final Exam; Exam duration: 2 hours; June, 8, 2017

Justify your answers and calculations

**1.** Consider a standard Brownian motion $B = \{B_t, t \geq 0\}$.

(a) Define the process $Y$ by

$$Y_t = B_t^2 + B_t^4 - t - 6\int_0^t B_s^2 \mathrm{ds}.$$

Is $Y$ a martingale with respect to the filtration generated by $B$? If it is, show it. If it is not, justify your conclusion.

(b) Let $\lambda > 0$ be a constant. Show that the process

$$Z_t = \lambda B_{\lambda^{-2}t}$$

is a standard Brownian motion.

**2.** Let $B = \{B_t, t \in [0, T]\}$ be a Brownian motion. Consider the process

$$L_t = \int_0^t f(r) dB_r,$$

where $f$ is a deterministic function such that $\int_0^T |f(s)|^2 ds < \infty$. Show that the characteristic function of $L_t$ is given by

$$\varphi(v) = \mathbb{E}\left[e^{ivL_t}\right] = exp\left(-\frac{v^2}{2}\int_0^t f(r)^2 dr\right),$$

and from this result deduce what is the distribution of $L_t$.

**3.** Let $B = \{B_t, t \in [0, T]\}$ be a Brownian motion. Consider a SDE of the type

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t,$$
$$X_0 = x,$$

where $f(t, x)$ and $c(t)$ are continuous and deterministic functions.

(a) If

$$f(t, x) = \frac{tx^5}{1 + x^4},$$

what can you say about the existence and uniqueness of solutions? Explain your answer.

(b) Considering $f(t, X_t) = e^{-t}X_t$ and $c(t) = t$, $X_0 = 1$, solve the SDE (Hint: perhaps it is useful to consider an appropriate integrating factor).

**4.** Consider the boundary value problem with domain $[0,T]\times\mathbb{R}$:

$$\begin{aligned}\frac{\partial F}{\partial t}+9x\frac{\partial F}{\partial x}+8x^2\frac{\partial^2 F}{\partial x^2}&=7F(t,x)\,,\quad t>0,\ x\in\mathbb{R}\\ F(T,x)&=x^4+\ln\left(x^2\right)\end{aligned}$$

Specify the infinitesimal generator of the associated diffusion, write the stochastic representation formula for the solution of the problem and obtain an expression for the solution of the boundary value problem.

**5.** Consider that you should propose a SDE for modeling the price $S_t$ of a risky financial asset (stock or index) and you know the historical values of the prices

$$S_0=2.50, S_1=2.67, S_2=2.64, ...$$

Moreover, suppose you have also estimates for the following parameters: the local mean rate of return ($k=0.08$), the rate of return for all the market ($\theta=0.06$), the volatility ($\sigma=0.15$), the risk-free interest rate ($r=0.05$) and the inflation rate ($\eta=0.02$).

(a) Which SDE do you propose and why? Which parameters are important for modeling the price $S_t$ and for the valuation of financial derivatives based on this underlying asset?

(b) What about the model based on the process $S_t=\mu t+\sigma B_t$, where $\mu$ is a constant and $B_t$ is a Brownian motion. What are your main comments and criticisms with respect to this model?

(c) Considering the Black-Scholes model, calculate the price (with no arbitrage) of the financial derivative with payoff $\chi=\Phi(S_T)=S_T^2+10\ln(S_T)$.

**6.** Let $B=\{B_t,t\geq 0\}$ be a Brownian motion. Consider the SDE

$$\begin{aligned}dX_t&=b(t,X_t)\,dt+\sigma(t,X_t)\,dB_t,\\ X_0&=x.\end{aligned}$$

Define the sequence $Y_t^{(n)}$ inductively, with $Y_t^{(0)}=X_0=x$ and

$$Y_t^{(n+1)}=x+\int_0^t b\left(s,Y_s^{(n)}\right)ds+\int_0^t\sigma\left(s,Y_s^{(n)}\right)dB_s,\quad n=1,2,....$$

Assume that

$$\mathbb{E}\left[\left|Y_t^{(n+1)}-Y_t^{(n)}\right|^2\right]\leq\frac{C^{n+1}t^{n+1}}{(n+1)!}.$$

Prove that the sequence $\left\{Y_t^{(n)}\right\}$ converges in quadratic mean (i.e., em $L^2(\mathbb{P})$) for a limit $X_t$, when $n\to\infty$ and that has a subsequence that converges a.s. (i.e., $Y_t^{(n_k)}(\omega)\to Y_t(\omega)$ a.s.)

Hint: You may start by proving that the sequence $\left\{Y_t^{(n)}\right\}$is a Cauchy sequence in $L^2(\mathbb{P})$.

Marks: 1(a):2.0, (b):2.0, 2:2.5, 3(a):2.0, (b):2.0, 4:2.5, 5(a):1.5, (b):1.5, (c):2.0, 6:2.0