Master in Mathematical Finance, ISEG, University of Lisbon

Stochastic Calculus

Final Exam; Exam duration: 2 hours; July, 3, 2017

Justify your answers and calculations

1. Consider a standard Brownian motion $B = \{B_t, t \geq 0\}$.

(a) Deduce for what values of the constants a and b is the process

$$Y_t = at^2 B_t - b \int_0^t sB_s \mathrm{ds}$$

a martingale with respect to the filtration generated by a Brownian motion

(b) Consider the process $Z_t = B_t + B_t^3$. Show that $\mathbb{E}[Z_t] = \mathbb{E}[Z_0] = 0 \quad \forall t \geq 0$. Does this implies that the process is a martingale? Explain your reasons.

2. Let $B = \{B_t, t \in [0,T]\}$ be a Brownian motion. Show that exists a process $u \in L^2_{a,T}$ such that

$$\exp(B_T) + 2B_T^2 + B_T = f(T) + \int_0^T u_t dB_t$$

and find explictly the process $u = \{u_t, t \in [0,T]\}$ and the deterministic function $f(T)$.

(Hint: consider the process $Y_t = \exp\left(B_t - \frac{t}{2}\right)$).

3. Consider the Brownian motion $B = \{B_t, t \in [0,T]\}$

(a) Let $Z = \{Z_t, t \in [0,T]\}$ be a stochastic process defined by

$$Z_t = e^{B_t}\left(1 + t^2\right).$$

Define $Y = \{Y_t, t \in [0,T]\}$ as the process with differential given by

$$dY_t = \frac{dZ_t}{Z_t}$$

and such that $Y_0 = 1$. Find explictly the process Y.

(b) Consider the stochastic differential equation

$$dX_t = \exp\left(-t^2\right) \ln\left(1 + X_t^4\right) dt + \frac{\cos(X_t)}{1+t^2} dB_t.$$
$$X_0 = L,$$

where L is a random variable such that $L \sim N(0,1)$ and L is independent of the Brownian motion B. Show that exists a unique solution for this stochastic differential equation.

4. Consider the boundary value problem with domain $[0,T] \times \mathbb{R}$:

$$\frac{\partial F}{\partial t} = 5F(t,x) - 9x\frac{\partial F}{\partial x} - 8x^2 \frac{\partial^2 F}{\partial x^2}, \quad t > 0, \ x \in \mathbb{R}$$
$$F(T,x) = 1 + x^4$$

Specify the infinitesimal generator of the associated diffusion, obtain an explicit expression for this diffusion process, write the stochastic representation formula for the solution of the problem and obtain an expression for the solution of the boundary value problem (as explicit as you can).

5. Consider the Black-Scholes model, with one risky asset S_t and one riskless asset B_t. Assets S_t and B_t have dynamics given by the SDE's

$$dS_t = \mu S_t\, dt + \sigma\, S_t\, d\overline{W}_t \quad \text{and} \quad dB_t = r\, B_t\, dt, \text{ with } B_0 = 1,$$

where $\overline{W}$ is a Brownian motion.

(a) Consider a self-financed portfolio $(\varphi(t), \psi(t))$ (of riskless assets and risky assets, respectively) and let V_t be the value of that portfolio at time t. Assuming that this portfolio replicates the value of a financial derivative with price given by $F\,(t, S_t)$ (i.e., $F(t, S_t) = V_t$), deduce that

$$\psi\,(t) = \frac{\partial F\,(t, S_t)}{\partial x}.$$

(Hint: apply the Itô formula to $F(t, S_t)$ and assume that the portfolio with value V_t is self-financed).

(b) Calculate the price (at time $t < T$) of the contingent claim with payoff

$$\chi = \Phi\,(S_T) = \begin{cases} \ln\,(2K) & \text{if } \ln\,(S_t) > 2K \\ \ln\,(K) & \text{if } K \leq \ln\,(S_t) \leq 2K \\ 0 & \text{if } \ln\,(S_t) < K \end{cases}$$

6. Let $B = \{B\,(t)\,, t \geq 0\}$ be a Brownian motion and define the process X by

$$X\,(t) = B(g(t)),$$

where $g(t)$ is a deterministic, differentiable function which is strictly increasing and $g(0) = 0$. Show that

$$\mathbb{E}\left[\sum_{i=0}^{n-1} |X\,(t_{i+1}) - X\,(t_i)|^2\right] = g(t),$$

for any partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, of $[0, t]$. Show also that

$$Var\left[\sum_{i=0}^{n-1} |X\,(t_{i+1}) - X\,(t_i)|^2\right] = 3\sum_{i=0}^{n-1} |g\,(t_{i+1}) - g\,(t_i)|^2 \longrightarrow 0,$$

when $n \to \infty$.

Marks: 1(a):2.25, (b):1.75, 2:2.25, 3(a):2.25, 3(b):2.5, 4:2.5, 5(a):2.0, (b):2.25, 6:2.25