LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Master in Actuarial Science
Rate Making and Experience Rating

Exam 2, 31/01/2018
Time allowed: 2:30

Instructions:

1. This paper contains 4 groups of questions and comprises 3 pages including the title page;
2. Enter all requested details on the cover sheet;
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so;
4. Number the pages of the paper where you are going to write your answers;
5. Attempt all questions;
6. Begin your answer to each of the 4 question groups on a new page;
7. Marks are shown in brackets. Total marks: 200;
8. Show calculations where appropriate;
9. An approved calculator may be used. No mobile phones or other communication devices are permitted;
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Suppose that given $\Theta = \theta$ the random variables $X_1, \ldots, X_n$ are independent with Poisson distribution with mean θ. Let $S = X_1 + \cdots + X_n$, $f_S(.)$ be the probability function of S, $\mathbf{X} = (X_1, \ldots, X_n)$, $\bar{X} = \sum_{j=1}^{n} X_j/n$, and $\pi(\theta)$ is the distribution of θ. **[85]**

   Bühlmann's credibility (pure) premium for the given risk, for the next year, is given by formula

$$P_c = z\bar{X} + (1 - z)\mu,$$

   where $z = n/(n + \upsilon/a)$, $\mu = E[\mu(\theta)]$, $\upsilon = E[\upsilon(\theta)]$, $a = V[\mu(\theta)]$; $\mu(\theta)$ and $\upsilon(\theta)$ are the risk mean and variance, respectively.

   (a) Find an expression for the probability function of S, $f_S(s)$. (10)]

   (b) Show that the Bayesian premium is given by

$$\mathbb{E}[X_{n+1}|\mathbf{X} = \mathbf{x}] = \frac{1+s}{n}\frac{f_S(1+s)}{f_S(s)},$$

   where $s = \sum_{i=1}^{n} x_i = n\bar{x}$. (15)

   (c) Consider now that $\pi(\theta)$ is exponential with mean 1. Calculate the distribution of S. (15)

   (d) Given that $n = 10$ and $\bar{x} = 0.1$, compute Bühlmann's credibility premium. (10)

   (e) Shortly, explain why, and which situations, we have *Credibility Exact.* (10)

   (f) Calculate $\mathbb{E}[P_c]$ and $\mathbb{V}[P_c]$ (10)

   (g) Explain the behaviour/variation of the credibility factor z as a function of υ or a, *ceteris paribus.* (15)

2. Suppose that claim sizes follow an exponential distribution with mean θ. For 80% of the risks we have $\theta = 8$ and for 20% of the risks we have $\theta = 2$. A policy was selected at random giving a claim size of 5 in year 1. **[35]**

   (a) Calculate the Bayesian premium. (15)

   (b) Calculate Bühlamn's credibility premium. (15)

   (c) If you have to choose one of the above estimates, which one would you choose. Explain. (5)

3. For a given motor insurance portfolioto, a certain insurer uses a *bonus-malus* system (BMS) to rate each individual risk . **[50]**

   (a) Consider a system with transition rules as represented in the matrices $[t_{ij}(k)]$ of Figure 1, $k = 0, 1, \ldots,$ where

$$t_{ij}(k) = \begin{cases} 1, & \text{if policy transfers from } i \text{ to } j \\ 0, & \textit{otherwise} \end{cases},$$

   if k claims arrive in the year.

$$\boldsymbol{T}(0) = \begin{pmatrix} 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{pmatrix}, \quad \boldsymbol{T}(1) = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$\boldsymbol{T}(2) = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{pmatrix} \text{ and } \boldsymbol{T}(k) = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{pmatrix} \text{ for all } k \geq 3.$$

Figure 1: Transition rules

   If number of claims is Poisson with mean 0.1, write the 1-step transition probability matrix.

   (b) Authors claim that a BMS based on credibility theory is

   ...optimal, fair and financial balanced.

However, that they also claim that

...it is not acceptable to regulators and managers.

i. Explain briefly the first quotation.

ii. Give 2-3 reasons explainig the second quotation.

(c) Write a table summarizing the transition rules of the system in Figure 1. Starting class is the highest level.

4. A working party is modelling a tariff for a given large motor insurance portfolio. [**30**]

The study group is proposing a new tariff for an existing portfolio evaluating a wide variety of commonly used risk factors that might have impact in both the claim frequency and the claim size means. Each risk factor may be divided into a short number of different levels.

(a) For the estimation of the pure premium, the means are analysed separately. Explain briefly.

(b) Suppose that the group is trying a Poisson model for the claim frequency considering three rating factors with three, two and two levels, respectively. They are labeled F11, F12, F13, F21, F22, F31 and F32, respectively. Write the estimated mean for the three rating factors with levels 2, 1 and 2, respectively.

(c) When modeling a tariff it is common to work with *key ratios* and *relativities*. Explain briefly why. You may use examples to illustrate.

(d) The insurer uses a *bonus-malus* system to charge each individual next year premium according to own current year claims. Comment the statement:

The bonus or malus levels should not be used as a risk factor for tariffication purpose.

Solutions:

1. (a) Given θ, $S|\theta \frown Poisson(n\theta)$, then we write

$$f_S(s) = \int_0^\infty \frac{(n\theta)^s e^{-n\theta}}{s!}\pi(\theta)d\theta\,.$$

(b) We can use the posterior distribution

$$\pi(\theta|\mathbf{x}) = \frac{\theta^{n\bar{x}}e^{-n\theta}\pi(\theta)}{\int_0^\infty \theta^{n\bar{x}}e^{-n\theta}\pi(\theta)d\theta}\,.$$

Then the Bayesian premium comes,

$$\int_0^\infty \theta\,\pi(\theta|\mathbf{x})d\theta = \frac{1+s}{n}\frac{f_S(1+s)}{f_S(s)}\,,$$

simplifying.

(c) We get a geometric distribution

$$f_S(s) = \frac{1}{1+n}\left(\frac{n}{1+n}\right)^s\,.$$

(d) This is an *exact credibility* situation, then we can write

$$\mathbb{E}[X_{n+1}|\mathbf{X}=\mathbf{x}] = \frac{1}{5}\frac{f_S(2)}{f_S(1)}\,.$$

(e) We get *credibility exact* when the Bayesian premium is linear on the observations, because in those situations credibility and Bayesian estimators are got minimizing the same loss function. That happens when we deal with conjugate distributions, so that prior and posterior are of the same family of distributions.

(f) We have $\mathbb{E}[P_c] = \mu$ and $\mathbb{V}[P_c] = z^2 Var[\bar{X}] = z^2(\nu/n + a) = a\,z$.

(g) If $k = \nu/a$ is big, when compared to n, then credibility is low. This means that a is small when compaerd to ν, there is small variability among risks. So, portfolio is very homogeneous, the collective premium must be trusted.

In the opposite way, high variability brings heterogeneity, you should take into more account the individually history. ...

2. (a)

$$\Pr(\theta = 8|X_1 = 5) = \frac{(1/8)e^{-5/8}(0.8)}{(1/8)e^{-5/8}(0.8) + (1/2)e^{-5/2}(0.2)} = 0.867035$$

$$E(X_2|X_1 05) = 0.867035\,(8) + 0.132965\,(2) = 7.202\,.$$

(b)

$$\begin{aligned}
\mu &= 0.8(8) + 0.2(2) = 6.8\\
\nu &= 0.8(64) + 0.2(4) = 52\\
a &= 0.8(64) + 0.2(4) - 6.8^2 = 5.76\\
z &= \frac{1}{(1 + 52/5.76)} = 0.0997230\\
P_c &= 0.0997230\,(5) + 0.900277\,(6.8) = 6.620499\,.
\end{aligned}$$

(c) This is **not** a case of *exact credibility.* Since the credibility premium makes the estimator to be linear, this is a an optimization problem subject to a constraint when compared to the Bayesian estimation. So, in general, the latter should be *better*...

3. (a)

$$\boldsymbol{P}(0.1) = \begin{pmatrix} 0.904837 & 0 & 0.090484 & 0 & 0.004524 & 0.000155 \\ 0.904837 & 0 & 0 & 0.090484 & 0 & 0.004679 \\ 0 & 0.904837 & 0 & 0 & 0.090484 & 0.004679 \\ 0 & 0 & 0.904837 & 0 & 0 & 0.095163 \\ 0 & 0 & 0 & 0.904837 & 0 & 0.095163 \\ 0 & 0 & 0 & 0 & 0.904837 & 0.095163 \end{pmatrix}$$

(b) i. Optimal: Results from minimizing the expected square loss function. Fair: Results from Bayes theorem. Balanced: On average we get the collective premium.

ii. It gives harsh penalties, may encourages uninsured driving, hit-and-run behaviour, leave company after one accident.

(c)

Starting level	Level occupied if 0	1	2	≥ 3 claim no. reported is
5	4	5	5	5
4	3	5	5	5
3	2	5	5	5
2	1	4	5	5
1	0	3	5	5
0	0	2	4	5

4. (a) Claim counts and severities are supposed to be independent then pure premium = mean frequency $\times$ severity mean... However one could model the aggregate mean using Tweedy models.

(b) e.g. $\exp\{intercept + \hat{F12} + \hat{F32}\}$, recall that for each factor we choose as much factor variables as the number of levels minus one. Above, we may consider the choice of discarding the first level in each rating factor.

(c) *Key ratios* are relative random variables, relative to exposure, e.g. capital insured, time period. A *relativity* is a factor multiplying to a base index quantity given as a unit. The premium will be calculated relatively to a base premium.

(d) Not necessarily. The use of a BMS may lead to a significant influence in the risk behaviour (like the *bonus hunger*) we must be cautious. The factor must be tested.