Course notes on ALM

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Abstract

1 Mean-Variance Analysis: Proofs

1.1 Preliminaries

This document gives all the proofs required for the different efficient portfolios in Mean Variance Analysis.

1.1.1 Differentiation of a quadratic form

Show that $\frac{\partial}{\partial \mathbf{w}} \mathbf{w}'\mathbf{A}\mathbf{w} = 2\mathbf{w}'\mathbf{A}$ if the matrix $\mathbf{A}^{n\times n}$ is symmetric and $\mathbf{w}^{n\times 1}$ is a vector.

Proof. If $\mathbf{w}$ is a vector and $f(\mathbf{w})$ a real-valued function, then $\frac{\partial}{\partial \mathbf{w}} f(\mathbf{w}) = \left(\frac{\partial}{\partial w_1} f(\mathbf{w}), \cdots, \frac{\partial}{\partial w_n} f(\mathbf{w})\right)$, also known as the gradient vector. The matrix product means $\mathbf{w}'\mathbf{A}\mathbf{w} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i a_{ij} w_j$, where

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Note that we require $a_{ij} = a_{ji}$ because $\mathbf{A}$ is supposed to be symmetric.

We do the proof for w_1. It works in the same way for the other w_i. The only summands involving w_1 are those where $i = 1$ and/or $j = 1$.

$$
\begin{aligned}
\frac{\partial}{\partial w_1}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i a_{ij} w_j &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial w_1}\left(w_i a_{ij} w_j\right) \\
&= \frac{\partial}{\partial w_1}\left(w_1 a_{11} w_1\right) \\
&\quad + \sum_{j=2}^{n}\frac{\partial}{\partial w_1}\left(w_i a_{1j} w_j\right) \text{ (rest of row 1)} \\
&\quad + \sum_{i=2}^{n}\frac{\partial}{\partial w_1}\left(w_i a_{i1} w_1\right) \text{ (rest of column 1)} \\
&= 2w_1 a_{11} + \sum_{j=2}^{n} a_{1j} w_j + \sum_{i=2}^{n} w_i a_{i1} \\
&= 2w_1 a_{11} + 2\sum_{i=2}^{n} w_i a_{i1} \text{ (use that } a_{ij} = a_{ji}) \\
&= 2\sum_{i=1}^{n} w_i a_{i1} = 2\mathbf{w}'\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1}\end{pmatrix}
\end{aligned}
$$

Thence we get

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{w}}\mathbf{w}'\mathbf{A}\mathbf{w} &= \left(\frac{\partial}{\partial w_1}\mathbf{w}'\mathbf{A}\mathbf{w}, \cdots, \frac{\partial}{\partial w_n}\mathbf{w}'\mathbf{A}\mathbf{w}\right) \\
&= \left(2\mathbf{w}'\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1}\end{pmatrix}, \cdots, 2\mathbf{w}'\begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn}\end{pmatrix}\right) \\
&= 2\mathbf{w}'\mathbf{A}.
\end{aligned}
$$

■

1.1.2 Lagrangian minimisation for finding a constrained minimum

When you want to minimize a differentiable function $f(x_1, \cdots, x_n)$ *without constraints*, you normally try to solve the equations

$$\frac{\partial}{\partial x_i} f(x_1, \cdots, x_n) \text{ for } i = 1, \cdots, n.$$

Assume now that you want to minimize a function $f(x_1, \cdots, x_n)$ *under the constraints*

$$g_1(x_1,\cdots,x_n)=0$$
$$g_2(x_1,\cdots,x_n)=0$$
$$\cdots$$
$$g_m(x_1,\cdots,x_n)=0$$

This can be done by defining a "Lagrange functional" or "Lagrangian"

$$L(x_1,\cdots,x_n;\lambda_1,\cdots,\lambda_m)=f(x_1,\cdots,x_n)-\lambda_1 g_1(x_1,\cdots,x_n)-\cdots-\lambda_m g_m(x_1,\cdots,x_n)$$

The variables $\lambda_1,\cdots,\lambda_m$ are called Lagrange multiplicators.
Then solve the equations

$$\begin{aligned}\frac{\partial}{\partial x_i}L(x_1,\cdots,x_n;\lambda_1,\cdots,\lambda_m)\text{ for } i &= 1,\cdots,n,\\ \frac{\partial}{\partial \lambda_j}L(x_1,\cdots,x_n;\lambda_1,\cdots,\lambda_m)\text{ for } j &= 1,\cdots,m.\end{aligned}$$

This produces (under suitable conditions) the constrained minimum of f.

1.2 Minimum variance portfolio

A very variance-averse investor could pose the asset allocation problem

$$\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\text{, subject to (only) }\mathbf{w}'\mathbf{1}=1$$

Using Lagrange minimisation, the optimal portfolio can be shown to be

$$\mathbf{w}_{\min}=\left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{1} \tag{1}$$

Its expected return is

$$\boldsymbol{\mu}'\mathbf{w}_{\min}=\left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}$$

and the variance of its return is

$$\mathbf{w}'_{\min}\boldsymbol{\Sigma}\mathbf{w}_{\min}=\left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}$$

Proof. The Lagrangian can be written as

$$L(\mathbf{w},\lambda)=\frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}-\lambda\left(\mathbf{w}'\mathbf{1}-1\right)$$

To determine $\mathbf{w}_{\min}$ we solve the linear equations

$$\frac{\partial}{\partial \mathbf{w}} L(\mathbf{w},\lambda) = \mathbf{w}'\boldsymbol{\Sigma} - \lambda \mathbf{1}' = \mathbf{0}', \tag{2}$$

$$\frac{\partial}{\partial \lambda} L(\mathbf{w},\lambda) = \mathbf{w}'\mathbf{1} - 1 = 0. \tag{3}$$

The first equation (2) gives us

$$\mathbf{w}' = \lambda \mathbf{1}'\boldsymbol{\Sigma}^{-1}$$

with λ to be determined. The second equation (3) then gives us

$$\lambda = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}$$

Therefore the solution is (after transposing $\mathbf{w}$ to be a column vector):

$$\mathbf{w} = \mathbf{w}_{\min} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1} \boldsymbol{\Sigma}^{-1}\mathbf{1}$$

This completes the proof. ■

1.3 Optimal portfolio of risky assets

A more demanding investor could pose the asset allocation problem

$$\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}, \text{ subject to } \mathbf{w}'\boldsymbol{\mu} = r \text{ and (of course) } \mathbf{w}'\mathbf{1} = 1$$

where r is the expected return that an allocation must provide in order to be a candidate.

The optimal portfolio $\mathbf{w}_r$ is now a linear combination of the minimum variance portfolio $\mathbf{w}_{\min}$ and one "reference" risky portfolio $\mathbf{w}_{\text{ref}}$:

$$\mathbf{w}_r = (1 - \upsilon)\, \mathbf{w}_{\min} + \upsilon \mathbf{w}_{\text{ref}} \tag{4}$$

The reference risky portfolio is

$$\mathbf{w}_{\text{ref}} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^{-1} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \tag{5}$$

or, in special cases, $\mathbf{w}_{\text{ref}} = \mathbf{w}_{\min} + \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$.

The weight of the risky portfolio in the optimal portfolio is

$$\upsilon = \upsilon(r) = \frac{r - \boldsymbol{\mu}'\mathbf{w}_{\min}}{\boldsymbol{\mu}'\mathbf{w}_{\text{ref}} - \boldsymbol{\mu}'\mathbf{w}_{\min}} \tag{6}$$

Thus the more return you ask for, the more risk you must accept.

Proof.

The Lagrangian can be written as

$$L(\mathbf{w},\lambda_1,\lambda_2) = \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \lambda_1 (\mathbf{w}'\mathbf{1} - 1) - \lambda_2 (\mathbf{w}'\boldsymbol{\mu} - r)$$

To determine $\mathbf{w}_r$ we solve the linear equations

$$\frac{\partial}{\partial \mathbf{w}} L(\mathbf{w},\lambda_1,\lambda_2) = \mathbf{w}'\boldsymbol{\Sigma} - \lambda_1 \mathbf{1}' - \lambda_2 \boldsymbol{\mu}' = \mathbf{0}', \tag{7}$$

$$\frac{\partial}{\partial \lambda_1} L(\mathbf{w},\lambda_1,\lambda_2) = \mathbf{w}'\mathbf{1} - 1 = 0, \tag{8}$$

$$\frac{\partial}{\partial \lambda_2} L(\mathbf{w},\lambda_1,\lambda_2) = \mathbf{w}'\boldsymbol{\mu} - r = 0. \tag{9}$$

Using (7) and using the definition (1) of $\mathbf{w}_{\min}$ we find that the solution $\mathbf{w}$ is of the general form

$$\mathbf{w} = \lambda_1 \boldsymbol{\Sigma}^{-1}\,\mathbf{1} + \lambda_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \lambda_1 \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right) \mathbf{w}_{\min} + \lambda_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}. \tag{10}$$

Inserting this into (8) we find that

$$\lambda_1 \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right) = 1 - \lambda_2 \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right). \tag{11}$$

Let us first consider the case where $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \neq 0$. In that case we can write(10) as

$$\begin{aligned}
\mathbf{w} &= \left(1 - \lambda_2 \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)\right) \mathbf{w}_{\min} + \lambda_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \\
&= \left(1 - \lambda_2 \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)\right) \mathbf{w}_{\min} + \lambda_2 \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right) \mathbf{w}_{\mathrm{ref}} \\
&= (1 - \upsilon)\,\mathbf{w}_{\min} + \upsilon \mathbf{w}_{\mathrm{ref}},
\end{aligned}$$

with a reference portfolio that is

$$\mathbf{w}_{\mathrm{ref}} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^{-1} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}.$$

We finally solve (9) to determine the weight to the reference portfolio

$$\upsilon = \upsilon(r) = \frac{r - \boldsymbol{\mu}'\mathbf{w}_{\min}}{\boldsymbol{\mu}'\mathbf{w}_{\mathrm{ref}} - \boldsymbol{\mu}'\mathbf{w}_{\min}}$$

This completes the proof the case where $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \neq 0$.
Let us now consider the case where $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = 0$. We use (11) to find

$$\lambda_1 = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}$$

and thence, using (10),

$$\mathbf{w}(\lambda_2) = \mathbf{w}_{\min} + \lambda_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}. \tag{12}$$

We then solve (9) to find

$$\boldsymbol{\mu}'\left(\mathbf{w}_{\min}+\lambda_2\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)=r\Rightarrow\lambda_2=\left(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^{-1}\left(r-\boldsymbol{\mu}'\mathbf{w}_{\min}\right)$$

so that (12) becomes

$$\mathbf{w}=\mathbf{w}_{\min}+\left(r-\boldsymbol{\mu}'\mathbf{w}_{\min}\right)\left(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

Now let $\mathbf{w}_{\mathrm{ref}}=\mathbf{w}_{\min}+\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and note that

$$\boldsymbol{\mu}'\mathbf{w}_{\mathrm{ref}}=\boldsymbol{\mu}'\mathbf{w}_{\min}+\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Rightarrow\boldsymbol{\mu}'\mathbf{w}_{\mathrm{ref}}-\boldsymbol{\mu}'\mathbf{w}_{\min}=\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

Therefore we can write

$$\begin{aligned}\mathbf{w}&=\mathbf{w}_{\min}+\frac{r-\boldsymbol{\mu}'\mathbf{w}_{\min}}{\boldsymbol{\mu}'\mathbf{w}_{\mathrm{ref}}-\boldsymbol{\mu}'\mathbf{w}_{\min}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\\&=\left(1-\frac{r-\boldsymbol{\mu}'\mathbf{w}_{\min}}{\boldsymbol{\mu}'\mathbf{w}_{\mathrm{ref}}-\boldsymbol{\mu}'\mathbf{w}_{\min}}\right)\mathbf{w}_{\min}+\frac{r-\boldsymbol{\mu}'\mathbf{w}_{\min}}{\boldsymbol{\mu}'\mathbf{w}_{\mathrm{ref}}-\boldsymbol{\mu}'\mathbf{w}_{\min}}\mathbf{w}_{\mathrm{ref}}\\&=\left(1-\upsilon(r)\right)\mathbf{w}_{\min}+\upsilon(r)\mathbf{w}_{\mathrm{ref}}.\end{aligned}$$

This completes the proof in the case of $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}=0$. ■

1.4 Minimum variance portfolio with a risk-free asset

We solve the problem

$$\min_{\mathbf{w}}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w},\text{ subject to }w_0+\mathbf{w}'\mathbf{1}=1$$

where w_0 is the allocation to the risk-free asset..
The optimal portfolio is (obviously)

$$w_0=1,\mathbf{w}=\mathbf{0}.$$

We prove this only to drill the technique.

Proof.

The Lagrangian can be written as

$$L(\mathbf{w},w_0,\lambda)=\frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}-\lambda\left(w_0+\mathbf{w}'\mathbf{1}-1\right)$$

To determine optimal portfolio we solve the linear equations

$$\frac{\partial}{\partial\mathbf{w}}L(\mathbf{w},w_0,\lambda)=\mathbf{w}'\boldsymbol{\Sigma}-\lambda\mathbf{1}'=\mathbf{0}',$$

$$\frac{\partial}{\partial w_0}L(\mathbf{w},w_0,\lambda)=\lambda=0,$$

$$\frac{\partial}{\partial\lambda}L(\mathbf{w},w_0,\lambda) = w_0 + \mathbf{w}'\mathbf{1} - 1 = 0.$$

The second equation gives immediately that $\lambda = 0$, the first equation thereupon gives that $\mathbf{w} = \mathbf{0}$, and finally the third equation gives that $w_0 = 1$.This completes the proof. ■

1.5 Optimal portfolio with a risk-free asset

Assume now that in addition to the n risky assets, you can invest in a risk-free asset ($i = 0$) that provides a secure return of $R_0 = \mu_0$.

Your asset allocation problem now becomes

$$\min_{w_0,\mathbf{w}} \mathbf{w}'\mathbf{\Sigma}\mathbf{w}, \text{ subject to } w_0\mu_0 + \mathbf{w}'\boldsymbol{\mu} = r \text{ and } w_0 + \mathbf{w}'\mathbf{1} = 1,$$

where r is the expected return that an allocation must provide in order to be a candidate, and w_0 is the proportion of your wealth to be invested risk-free.

In this case, the optimal portfolio is a combination of

1. a risk-free investment of w_0, and
2. investment of the remaining $1 - w_0$ in a tangency portfolio $\mathbf{w}_{\tan}$.

The relevant parameters are

$$\mathbf{w}_{\tan} = \left(\mathbf{1}'\mathbf{\Sigma}^{-1}(\boldsymbol{\mu}-\mu_0\mathbf{1})\right)^{-1}\mathbf{\Sigma}^{-1}(\boldsymbol{\mu}-\mu_0\mathbf{1})$$

$$1 - w_0 = \frac{r-\mu_0}{\boldsymbol{\mu}'\mathbf{w}_{\tan} - \mu_0}$$

We assume that $\mathbf{1}'\mathbf{\Sigma}^{-1}(\boldsymbol{\mu}-\mu_0\mathbf{1}) \neq 0$.

Proof. The Lagrangian can be written as

$$L(w_0,\mathbf{w},\lambda_1,\lambda_2) = \frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} - \lambda_1(w_0 + \mathbf{w}'\mathbf{1} - 1) - \lambda_2(w_0\mu_0 + \mathbf{w}'\boldsymbol{\mu} - r)$$

To determine the optimal $(w_0,\mathbf{w})$ we solve the linear equations

$$(\partial/\partial\mathbf{w})\, L(w_0,\mathbf{w},\lambda_1,\lambda_2) = \mathbf{w}'\,\mathbf{\Sigma} - \lambda_1\mathbf{1}' - \lambda_2\boldsymbol{\mu}' = \mathbf{0}' \qquad (13)$$

$$(\partial/\partial w_0)\, L(w_0,\mathbf{w},\lambda_1,\lambda_2) = -\lambda_1 - \lambda_2\mu_0 = 0 \qquad (14)$$

$$(\partial/\partial\lambda_1)\, L(w_0,\mathbf{w},\lambda_1,\lambda_2) = w_0 + \mathbf{w}'\mathbf{1} - 1 = 0 \qquad (15)$$

$$(\partial/\partial\lambda_2)\, L(w_0,\mathbf{w},\lambda_1,\lambda_2) = w_0\mu_0 + \mathbf{w}'\boldsymbol{\mu} - r = 0 \qquad (16)$$

Using (13) we find that the solution $\mathbf{w}$ is of the general form

$$\mathbf{w} = \lambda_1 \boldsymbol{\Sigma}^{-1}\mathbf{1} + \lambda_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}.$$

Using (14) we find that $\lambda_1 = -\lambda_2\mu_0$, so that

$$\mathbf{w} = \lambda_2 \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \mu_0\mathbf{1}\right)$$

Using (15) we find

$$\lambda_2 = \left(1 - w_0\right) / \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \mu_0\mathbf{1}\right)\right)$$

so that

$$\mathbf{w} = \left(1 - w_0\right)\left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \mu_0\mathbf{1}\right)\right)^{-1}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \mu_0\mathbf{1}\right) = \left(1 - w_0\right)\mathbf{w}_{\text{tan}}.$$

Finally, (16) gives us

$$1 - w_0 = \frac{r - \mu_0}{\boldsymbol{\mu}'\mathbf{w}_{\text{tan}} - \mu_0}$$

Note that the tangency portfolio is a function of the available risk-free return. The variance of the overall return is

$$\sigma^2(r) = \text{Var}\left(w_0\mu_0 + \left(1 - w_0\right)\mathbf{w}_{\text{tan}}'\mathbf{R}\right) = \left(1 - w_0\right)^2 \mathbf{w}_{\text{tan}}'\boldsymbol{\Sigma}\mathbf{w}_{\text{tan}}$$

This completes the proof.

■

1.6 Minimum surplus variance portfolio

Let us assume that there are n investible assets with a random return characterised by its mean vector and covariance matrix:

$$\mathbf{R} \sim \left[\boldsymbol{\mu}, \boldsymbol{\Sigma}\right]$$

We make the additional assumption that liability growth is random, and correlated with asset returns:

$$\begin{aligned}
\text{E}\left(R_L\right) &= \mu_L \\
\text{Var}\left(R_L\right) &= \sigma_L^2 \\
\text{Cov}\left(R_i, R_L\right) &= \gamma_{i,L} = \rho_{i,L}\sigma_i\sigma_L
\end{aligned}$$

Denote the vector of covariances by

$$\boldsymbol{\gamma} = \left(\gamma_{1,L}, \cdots, \gamma_{n,L}\right)'$$

and assume that you know (have estimated) μ_L, σ_L^2 and $\boldsymbol{\gamma}$. Let F denote the initial funding ratio, $F = W(0)/L(0)$.

With an arbitrary asset allocation vector $\mathbf{w}$, the random surplus return is

$$R_S = \mathbf{w}'\mathbf{R} - \frac{R_L}{F}$$

It is easy to verify that

$$\begin{aligned} \mathrm{E}(R_S) &= \mathbf{w}'\boldsymbol{\mu} - \frac{\mu_L}{F} \\ \mathrm{Var}(R_S) &= \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\boldsymbol{\gamma}}{F} \end{aligned}$$

Let us minimise the variance, subject to constraints.

If your only aim is to minimise variance, you solve:

$$\min_{\mathbf{w}} \left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\boldsymbol{\gamma}}{F} \right) \text{ subject to } \mathbf{w}'\mathbf{1} = 1$$

The optimal portfolio is

$$\mathbf{w}_{\min}(F, \boldsymbol{\gamma}) = (1 - \upsilon)\,\mathbf{w}_{\min} + \upsilon \mathbf{w}_{\boldsymbol{\gamma}},$$

where $\mathbf{w}_{\min}$ is the unconditional minimum variance allocation and $\mathbf{w}_{\boldsymbol{\gamma}}$ is the *liability hedge portfolio*. The liability hedge portfolio is

$$\mathbf{w}_{\boldsymbol{\gamma}} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\right)^{-1} \boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}$$

The weight of the liability hedge portfolio in the optimal portfolio is

$$\upsilon = \upsilon(F, \boldsymbol{\gamma}) = \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}$$

In the case of $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} = 0$, we can write $\mathbf{w}_{\boldsymbol{\gamma}} = \mathbf{w}_{\min} + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}$.

Proof. The Lagrangian can be written as

$$L(\mathbf{w}, \lambda) = \frac{1}{2}\left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\boldsymbol{\gamma}}{F} \right) - \lambda(\mathbf{w}'\mathbf{1} - 1).$$

To determine $\mathbf{w}$ we solve the linear equations

$$\frac{\partial}{\partial \mathbf{w}} L(\mathbf{w}, \lambda) = \mathbf{w}'\,\boldsymbol{\Sigma} - \frac{1}{F}\boldsymbol{\gamma}' - \lambda\mathbf{1}' = \mathbf{0}', \tag{17}$$

$$\frac{\partial}{\partial \lambda} L(\mathbf{w}, \lambda) = \mathbf{w}'\mathbf{1} - 1 = 0. \tag{18}$$

Equation (17) gives

$$\mathbf{w} = \lambda\boldsymbol{\Sigma}^{-1}\mathbf{1} + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}. \tag{19}$$

Substituting this in (18) and solving for λ gives

$$\lambda = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}\left(1 - \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\right).$$

We insert this back into (19) gives

$$\begin{aligned}\mathbf{w} &= \lambda\boldsymbol{\Sigma}^{-1}\mathbf{1} + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \\ &= \left(1 - \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\right)\left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{1} + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\end{aligned}$$

If $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \neq 0$, we can write this as

$$\begin{aligned}\mathbf{w} &= \left(1 - \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\right)\mathbf{w}_{\min} + \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\right)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \\ &= (1-\upsilon)\,\mathbf{w}_{\min} + \upsilon\mathbf{w}_{\boldsymbol{\gamma}}\end{aligned}$$

If $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} = 0$, we can write

$$\mathbf{w} = \mathbf{w}_{\min} + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}$$

This completes the proof. ■

1.7 Optimal asset allocation to fund a stochastic liability, optimal portfolio of risky assets

If you are more interested in beating than in meeting the expected return of the liability hedge portfolio, you would solve:

$$\min_{\mathbf{w}}\left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\boldsymbol{\gamma}}{F}\right) \text{ subject to } \mathbf{w}'\boldsymbol{\mu} = r \text{ and } \mathbf{w}'\mathbf{1} = 1$$

where r is the expected return that an asset allocation must provide in order to be a candidate for you.

The additional constraint only makes sense if $r \geq \boldsymbol{\mu}'\mathbf{w}_{\min}(F, \boldsymbol{\gamma})$.

The optimal portfolio can be written in the form

$$\begin{aligned}\mathbf{w}_r(F,\boldsymbol{\gamma}) &= (1-\upsilon-\omega)\,\mathbf{w}_{\min} + \omega\mathbf{w}_{\text{ref}} + \upsilon\mathbf{w}_{\boldsymbol{\gamma}} \\ &= \mathbf{w}_{\min}(F,\boldsymbol{\gamma}) + \omega\left(\mathbf{w}_{\text{ref}} - \mathbf{w}_{\min}\right),\end{aligned}$$

where

- $\mathbf{w}_{\min}$ denotes the unconditional minimum variance allocation,
- $\mathbf{w}_{\text{ref}}$ the risky reference portfolio when there is no risk-free asset,

- $\mathbf{w}_{\boldsymbol{\gamma}}$ the liability hedge portfolio, and
- $\mathbf{w}_{\min}(F, \boldsymbol{\gamma})$ the minimum surplus variance allocation.

The weighting parameters are

$$\upsilon = \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \text{ and } \omega = \frac{r - \boldsymbol{\mu}'\mathbf{w}_{\min}(F, \boldsymbol{\gamma})}{\boldsymbol{\mu}'\mathbf{w}_{\text{ref}} - \boldsymbol{\mu}'\mathbf{w}_{\min}}.$$

Proof. The Lagrangian can be written as

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\boldsymbol{\gamma}}{F}\right) - \lambda_1(\mathbf{w}'\mathbf{1} - 1) - \lambda_2(\mathbf{w}'\boldsymbol{\mu} - r).$$

To determine $\mathbf{w}$ we solve the linear equations

$$(\partial/\partial\mathbf{w})\, L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\,\boldsymbol{\Sigma} - \frac{1}{F}\boldsymbol{\gamma}' - \lambda_1\mathbf{1}' - \lambda_2\boldsymbol{\mu}' = \mathbf{0}', \tag{20}$$

$$(\partial/\partial\lambda_1)\, L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mathbf{1} - 1 = 0, \tag{21}$$

$$(\partial/\partial\lambda_2)\, L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\boldsymbol{\mu} - r = 0. \tag{22}$$

In what follows we assume that all quantities that are divided by, are non-zero.

From (20) we obtain

$$\mathbf{w} = \lambda_1\boldsymbol{\Sigma}^{-1}\mathbf{1} + \boldsymbol{\lambda_2}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \tag{23}$$

Inserting this in (21) and solving for λ_1 we obtain

$$\lambda_1 = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}\left(1 - \boldsymbol{\lambda_2}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\right)$$

Substituting this expression for λ_1 in (23) gives

$$\begin{aligned}
\mathbf{w} &= \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}\left(1 - \boldsymbol{\lambda_2}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\right)\boldsymbol{\Sigma}^{-1}\mathbf{1} + \boldsymbol{\lambda_2}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \qquad (24)\\
&= \mathbf{w}_{\min} + \boldsymbol{\lambda_2}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{w}_{\min}\right) + \left(\frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} - \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\mathbf{w}_{\min}\right)\\
&= \mathbf{w}_{\min} + \boldsymbol{\lambda_2}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\left(\mathbf{w}_{\text{ref}} - \mathbf{w}_{\min}\right) + \upsilon\left(\mathbf{w}_{\boldsymbol{\gamma}} - \mathbf{w}_{\min}\right)\\
&= \mathbf{w}_{\min}(F, \boldsymbol{\gamma}) + \boldsymbol{\lambda_2}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\left(\mathbf{w}_{\text{ref}} - \mathbf{w}_{\min}\right),
\end{aligned}$$

where $\upsilon = \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}$ and $\mathbf{w}_{\min}(F, \boldsymbol{\gamma}) = (1 - \upsilon)\,\mathbf{w}_{\min} + \upsilon\mathbf{w}_{\boldsymbol{\gamma}}$.

Inserting this in (22) and solving for λ_2 we obtain

$$
\begin{aligned}
\lambda_2 &= \left(\mathbf{1}'\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{\mu}\right)^{-1}\left(\frac{r-\boldsymbol{\mu}'\mathbf{w}_{\min}(F,\boldsymbol{\gamma})}{\boldsymbol{\mu}'\mathbf{w}_{\text{ref}}-\boldsymbol{\mu}'\mathbf{w}_{\min}}\right) \\
&= \left(\mathbf{1}'\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{\mu}\right)^{-1}\omega,
\end{aligned}
$$

where $\omega = \frac{r-\boldsymbol{\mu}'\mathbf{w}_{\min}(F,\boldsymbol{\gamma})}{\boldsymbol{\mu}'\mathbf{w}_{\text{ref}}-\boldsymbol{\mu}'\mathbf{w}_{\min}}$.
Inserting this expression for λ_2 in the last expression of (24), we finally find

$$
\begin{aligned}
\mathbf{w} &= \mathbf{w}_{\min}(F,\boldsymbol{\gamma})+\omega\left(\mathbf{w}_{\text{ref}}-\mathbf{w}_{\min}\right) \\
&= (1-\upsilon-\omega)\,\mathbf{w}_{\min}+\omega\mathbf{w}_{\text{ref}}+\upsilon\mathbf{w}_{\boldsymbol{\gamma}}
\end{aligned}
$$

This completes the proof ■

1.8 Optimal asset allocation to fund a stochastic liability, optimal portfolio including a risk-free asset

Let us finally develop the case where the investor has access to a risk-free asset with secure return μ_0. The problem is then to

$$
\min_{w_0,\mathbf{w}}\left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}+\frac{\sigma_L^2}{F^2}-2\frac{\mathbf{w}'\boldsymbol{\gamma}}{F}\right) \text{ subject to } w_0\mu_0+\mathbf{w}'\boldsymbol{\mu}=r \text{ and } w_0+\mathbf{w}'\mathbf{1}=1
$$

The parameter w_0 denotes the proportion of assets invested risk-free.
The optimal portfolio consists of

- a risk-free investment of w_0,
- investment of $1-w_0-\upsilon$ in the tangency portfolio $\mathbf{w}_{\tan}$,
- investment of υ in the liability hedge portfolio $\mathbf{w}_{\boldsymbol{\gamma}}$.

The weightings are

$$
\upsilon=\frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \text{ and } 1-w_0=\frac{r-\upsilon\left(\boldsymbol{\mu}'\mathbf{w}_{\boldsymbol{\gamma}}-\boldsymbol{\mu}'\mathbf{w}_{\tan}\right)-\mu_0}{\boldsymbol{\mu}'\mathbf{w}_{\tan}-\mu_0}.
$$

Proof. As always we start with the Lagrangian

$$
L(\mathbf{w},w_0,\lambda_1,\lambda_2)=\frac{1}{2}\left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}+\frac{\sigma_L^2}{F^2}-2\frac{\mathbf{w}'\boldsymbol{\gamma}}{F}\right)-\lambda_1(w_0+\mathbf{w}'\mathbf{1}-1)-\lambda_2(w_0\mu_0+\mathbf{w}'\boldsymbol{\mu}-r)\,.
$$

Its derivative that we need to equate to zero, are

$$
(\partial/\partial\mathbf{w})\,L(\mathbf{w},w_0,\lambda_1,\lambda_2)=\mathbf{w}'\,\boldsymbol{\Sigma}-\frac{1}{F}\boldsymbol{\gamma}'-\lambda_1\mathbf{1}'-\lambda_2\boldsymbol{\mu}'=\mathbf{0}', \qquad (25)
$$

$$(\partial/\partial w_0)\, L(\mathbf{w}, w_0, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2 \mu_0 = 0 \tag{26}$$

$$(\partial/\partial \lambda_1)\, L(\mathbf{w}, w_0, \lambda_1, \lambda_2) = w_0 + \mathbf{w}'\mathbf{1} - 1 = 0, \tag{27}$$

$$(\partial/\partial \lambda_2)\, L(\mathbf{w}, w_0, \lambda_1, \lambda_2) = w_0 \mu_0 + \mathbf{w}'\boldsymbol{\mu} - r = 0. \tag{28}$$

In what follows we assume that all quantities that are divided by, are non-zero.

From (25) we obtain

$$\mathbf{w} = \lambda_1 \boldsymbol{\Sigma}^{-1}\mathbf{1} + \lambda_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}. \tag{29}$$

From (26) we obtain

$$\lambda_1 = -\lambda_2 \mu_0$$

Insert this in (29) and transform to

$$\mathbf{w} = \lambda_2 \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mu_0 \mathbf{1}) + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \tag{30}$$

Insert this expression for $\mathbf{w}$ into (27) to find

$$\begin{aligned} 1 &= w_0 + \mathbf{1}'\mathbf{w} = w_0 + \lambda_2 \mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mu_0 \mathbf{1}) + \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \\ &\Rightarrow \\ \lambda_2 &= \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mu_0 \mathbf{1})\right)^{-1}(1 - w_0 - \upsilon) \end{aligned}$$

Therefore (30) can be written as

$$\begin{aligned} \mathbf{w} &= \lambda_2 \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mu_0 \mathbf{1}) + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma} \\ &= (1 - w_0 - \upsilon)\,\mathbf{w}_{\tan} + \upsilon \mathbf{w}_{\boldsymbol{\gamma}} \end{aligned}$$

where $\mathbf{w}_{\tan} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mu_0 \mathbf{1})\right)^{-1} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mu_0 \mathbf{1})$ and $\mathbf{w}_{\boldsymbol{\gamma}} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\right)^{-1} \boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}$. Now it only remains to determine w_0. We do this by solving (28):

$$\begin{aligned} r &= w_0 \mu_0 + \boldsymbol{\mu}'\mathbf{w} \\ &= w_0 \mu_0 + (1 - w_0 - \upsilon)\,\boldsymbol{\mu}'\mathbf{w}_{\tan} + \upsilon \boldsymbol{\mu}'\mathbf{w}_{\boldsymbol{\gamma}} \\ &= w_0 (\mu_0 - \boldsymbol{\mu}'\mathbf{w}_{\tan}) + \boldsymbol{\mu}'\mathbf{w}_{\tan} + \upsilon\left(\boldsymbol{\mu}'\mathbf{w}_{\boldsymbol{\gamma}} - \boldsymbol{\mu}'\mathbf{w}_{\tan}\right). \end{aligned}$$

Thus

$$
\begin{aligned}
1 - w_0 &= 1 - \frac{r - \boldsymbol{\mu}'\mathbf{w}_{\tan} - \upsilon\left(\boldsymbol{\mu}'\mathbf{w}_{\boldsymbol{\gamma}} - \boldsymbol{\mu}'\mathbf{w}_{\tan}\right)}{\mu_0 - \boldsymbol{\mu}'\mathbf{w}_{\tan}} \\
&= \frac{\mu_0 - \boldsymbol{\mu}'\mathbf{w}_{\tan} - r + \boldsymbol{\mu}'\mathbf{w}_{\tan} + \upsilon\left(\boldsymbol{\mu}'\mathbf{w}_{\boldsymbol{\gamma}} - \boldsymbol{\mu}'\mathbf{w}_{\tan}\right)}{\mu_0 - \boldsymbol{\mu}'\mathbf{w}_{\tan}} \\
&= \frac{r - \upsilon\left(\boldsymbol{\mu}'\mathbf{w}_{\boldsymbol{\gamma}} - \boldsymbol{\mu}'\mathbf{w}_{\tan}\right) - \mu_0}{\boldsymbol{\mu}'\mathbf{w}_{\tan} - \mu_0}.
\end{aligned}
$$

This completes the proof. ■