3 – Hedging Interest Rate Risk

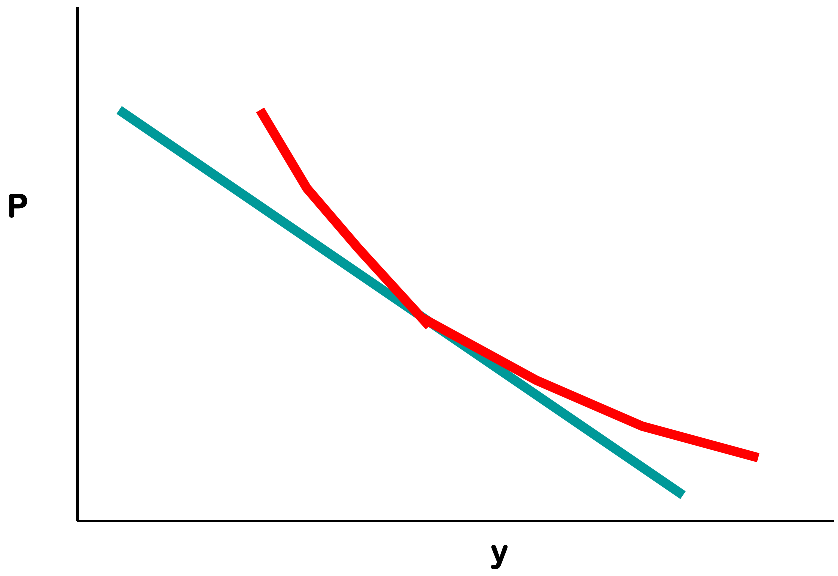
- Basic principle: attempt to reduce as much as possible the dimensionality of the problem, i.e. to hedge risk with as few factors as possible.

- First step: **duration hedging**
  - Consider only one risk factor
  - Assume only small changes in the risk factor

- Beyond duration: **convexity hedging**
  - Relax the assumption of small interest rate changes

3.1 - DURATION

- We will study the sensitivity of the bond price to changes in yield - Interest rate risk:
  - Rates change from *y* to *y+dy*
  - *dy* - small variation in yields, e.g. 1 bp (e.g., from 5% to 5.01%)
  - *dP* - variation in bond price due to *dy*
- The relationship between the bond prices and the yields is not linear.
- However, for small changes in yields, a good proxy for *dP* is the first derivative of the bond price in order to *y*.



- With **continuously compounded interest rates** and assuming a flat yield curve (same yields for all maturities), we have:

$$P^c = FVe^{-yT} + \sum_{n=1}^{T} ce^{-yn}$$

$$\frac{\partial P^c}{\partial y} = \frac{\partial [FVe^{-yT} + \sum_{n=1}^{T} ce^{-yn}]}{\partial y} = -T \cdot FVe^{-yT} - \sum_{n=1}^{T} n \cdot ce^{-yn}$$

- **Macaulay Duration (Frederick Macaulay, 1938) –** aka effective maturity: **Average maturity (measured in years) of all cash-flows weighted by the relevance of their NPV on the bond price** (while residual maturity is just the maturity of the final cash-flow), assuming the yield curve is flat.

⇓

Macaulay Duration > 0

- Non-flat yield curve => Fisher-Weil Duration

- Calculated as (the absolute value of) the partial derivative of the bond price with respect to yield, divided by the bond price:

$$D = \frac{\sum_{n=1}^{T} n \cdot ce^{-yn} + T \cdot FVe^{-yT}}{P^c}$$
$$= 1 \cdot \frac{ce^{-y}}{P^c} + 2 \cdot \frac{ce^{-2y}}{P^c} + 3 \cdot \frac{ce^{-3y}}{P^c} + \cdots + T \cdot \frac{ce^{-yT}}{P^c} + T \cdot \frac{FVe^{-yT}}{P^c}$$

$$\frac{\partial P^c}{\partial y} = -DP^c \Rightarrow \frac{dP^c}{P^c} = -Ddy$$

Duration corresponds to the relative (%) change in price due to a small change in yield

- With **discrete compounding interest rates**:

$$P^c = \frac{FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c}{(1+y)^n}$$

$$\frac{\partial P^c}{\partial y} = \frac{\partial \left[\frac{FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c}{(1+y)^n} \right]}{\partial y}$$

$$= -\frac{T \cdot FV(1+y)^{T-1}}{(1+y)^{2T}} - \sum_{n=1}^{T} \frac{c \cdot n(1+y)^{n-1}}{(1+y)^{2n}}$$

$$= -\frac{T \cdot FV}{(1+y)^{T+1}} - \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^{n+1}}$$

$$= -\frac{1}{(1+y)} \left[\frac{T \cdot FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^n} \right]$$

Weighted-average maturity of all cash-flows (weighted by the relative weight of their NPV on the bond price) ←

$$D = \frac{\frac{T \cdot FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^n}}{P^c}$$

$$\frac{\partial P^c}{\partial y} = -\frac{1}{(1+y)} D P^c \Rightarrow \frac{dP^c}{P^c} = -\frac{1}{(1+y)} D dy$$

$$\frac{1}{(1+y)} D = MD$$ → **modified duration** – percentage impact (%) on bond price of a given change (percentage points) in the yield

EXAMPLE (SEE CALCULATION IN SPREADSHEET)

T = 10, *c* = 5%, *y* = 5% (bond at par)

Time of Cash Flow	Cash Flow F_n	$w_n = \frac{1}{P^C} \cdot \frac{F_n}{(1+y)^n}$	$n \cdot w_n$	$n^2 \cdot ce^{-yn}$
1	50	0,047619048	0,047619	47,56147123
2	50	0,045351474	0,090703	180,9674836
3	50	0,04319188	0,129576	387,3185894
4	50	0,041135124	0,16454	654,9846025
5	50	0,039176308	0,195882	973,5009788
6	50	0,03731077	0,223865	1333,472797
7	50	0,035534067	0,248738	1726,48582
8	50	0,033841968	0,270736	2145,024147
9	50	0,032230446	0,290074	2582,394014
10	1050	0,644608916	6,446089	3032,653299
Total			8,107822	13064,3632

$$D = \frac{\frac{T \cdot FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^n}}{P^c} \cong 8$$

- The lower the coupon rate, the higher (and closer to residual maturity) the duration will be, as the relative weight of the final cash-flow will be higher => zero-coupon bonds have duration equal to residual maturity.
- For a given coupon rate and yield, duration increases as maturity increases: $\frac{\partial D}{\partial n} \geq 0$
- For a given maturity and coupon rate, duration increases as the yield decreases, as the net present value of the cash-flows increase more in longer than in shorter maturities: $\frac{\partial D}{\partial y} \leq 0$

DURATION HEDGING

- Principle: immunize the value of a bond portfolio with respect to changes in yield:
  - P = value of the portfolio
  - H = value of the hedging instrument
- Hedging instrument may be:
  - Bonds
  - Swaps
  - Futures
  - Options
  - FRAs
- Duration hedging is very simple to do, but it is only valid for small changes and parallel shifts of the yield curve.

- Changes in value
  - Portfolio $$dP \approx P'(y)dy$$
  - Hedging instrument $$dH \approx H'(y)dy$$

- Strategy: hold *q* units of the hedging instrument so that

$$dP + qdH = \left(qH'(y) + P'(y)\right)dy = 0$$

- Solution

$$q = -\frac{P'(y)}{H'(y)} = -\frac{P \times D_P}{H \times D_H}$$

given that P'(y)=dP/dY and dP/P = -D x dy (with countinuously compounded interest rates) ⇔dP/dy = -P x D

Example:

- At date t, a portfolio P has a price of $328635, a 5.143% yield and a 7.108 duration.

- Hedging instrument – bond with price = $118.786, yield = 4.779% and duration = 5.748.

- Hedging strategy involves taking a short position (i.e. selling futures contracts) as follows:

$$q = -\frac{P'(y)}{H'(y)} = -\frac{P \times D_P}{H \times D_H}$$

q = -(328635x7.108)/(118.786x5.748) = - 3421

- Therefore, 3421 units of the hedging bond should be sold.

3.2. CONVEXITY

- Considering a second order Taylor approximation:

$$\frac{dP}{P} = \underbrace{\frac{dP}{dy}\frac{1}{P}}_{-D}(dy) + \frac{1}{2}\underbrace{\frac{d^2P}{dy^2}\frac{1}{P}}_{C}(dy)^2$$

- With continuous compounding:

$$\frac{\partial^2 P}{\partial y^2} = T^2 \cdot FVe^{-yT} + \sum_{n=1}^{T} n^2 \cdot ce^{-yn} \quad \text{as} \quad \frac{\partial P}{\partial y} = -T \cdot FVe^{-yT} - \sum_{n=1}^{T} n \cdot ce^{-yn}$$

$$C = \frac{\partial^2 P}{\partial y^2} \cdot \frac{1}{P} = \frac{T^2 \cdot FVe^{-yT} + \sum_{n=1}^{T} n^2 \cdot ce^{-yn}}{P} \geq 0$$

- With discrete compounding, convexity may be written as a function of MD and its first derivative in order to the yield:

$$C = \frac{\partial^2 P}{\partial y^2} \cdot \frac{1}{P} \leftrightarrow CP = \frac{\partial^2 P}{\partial y^2}$$

$$\frac{\partial P}{\partial y} = -MD \cdot P \Rightarrow CP = \frac{\partial(-MD \cdot P)}{\partial y} = \frac{\partial(-MD)}{\partial y} \cdot P + \frac{\partial P}{\partial y} \cdot (-MD)$$

$$= \frac{\partial(-MD)}{\partial y} \cdot P + \left(-\frac{1}{(1+y)} DP\right) \cdot (-MD)$$

$$= -\frac{\partial(MD)}{\partial y} \cdot P + \underbrace{(-MD \cdot P)}_{} \cdot (-MD) \leftrightarrow$$

$$\leftrightarrow C = -\frac{\partial(MD)}{\partial y} + MD^2$$

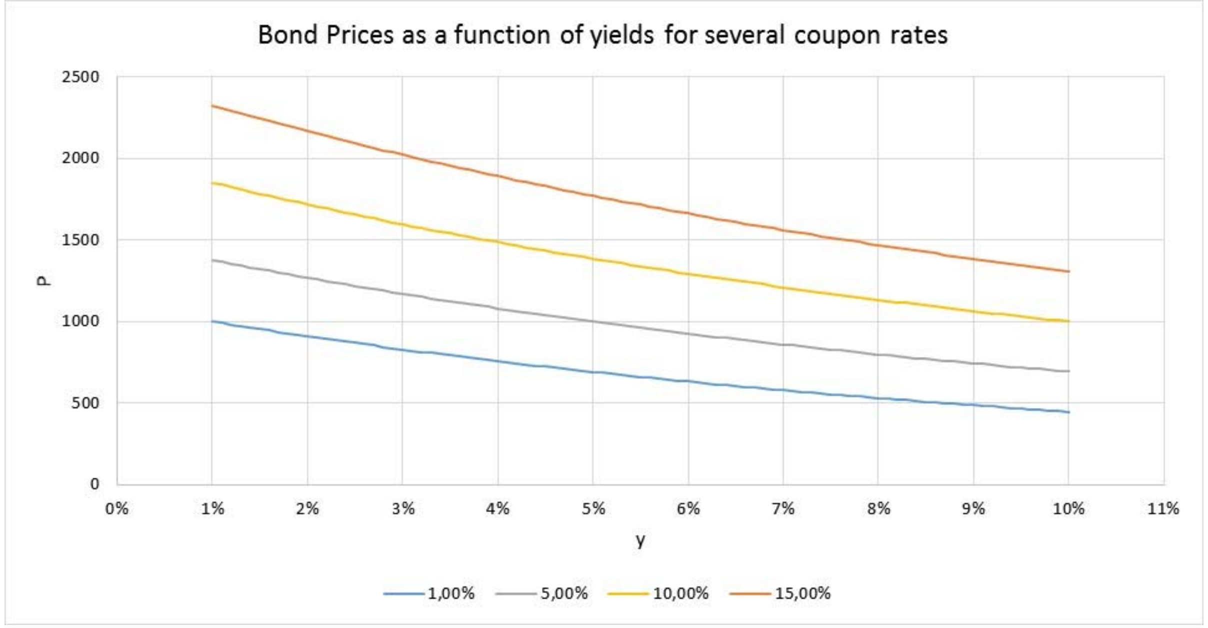
- As MD decreases with the yield => $C \geq 0$

unless bonds have embedded options, namely the prepayment option (Call−option hold by the issuer), as higher interest rates => higher prepayments => lower duration (negative convexity)

CONVEXITY- PROPERTIES

$$C = \frac{\partial^2 P}{\partial y^2} \cdot \frac{1}{P} = \frac{T^2 \cdot FVe^{-yT} + \sum_{n=1}^{T} n^2 \cdot ce^{-yn}}{P} \geq 0$$

- For a given maturity and yield, convexity increases when the bond provides regular payments along time => convexity increases with the coupon rate.
- But if the coupon rate increases, the yield will also increase, bring convexity (and duration) down.
- For a given maturity and coupon rate, convexity increases when the yield decreases.
- For a given coupon rate and yield, convexity increases when maturity increases.
- A bond with higher convexity is always preferred, as its price benefits more from yield decreases and its less impacted by yield increases => bonds with low coupon rates.



DURATION + CONVEXITY HEDGING - PRINCIPLE

- Principle: immunize the value of a bond portfolio with respect to changes in yield:
  - Denote by *P* the value of the portfolio
  - 2 hedging instruments (whose value is H_1 and H_2) because there are 2 risk factors to be hedged - parallel shifts in the yield curve and the second order effect:
  - Portfolio value variations: $$dP \approx P'(y)dy + \frac{P''(y)}{2}dy^2$$
  - Hedging instruments value variations: $$\begin{cases} dH_1 \approx H_1'(y)dy + \frac{1}{2}H_1''(y)dy^2 \\ dH_2 \approx H_2'(y)dy + \frac{1}{2}H_2''(y)dy^2 \end{cases}$$

- Strategy: hold q_1 and q_2 units of the 1st and 2nd hedging instruments so that:

$$dP + q_1 \times dH_1 + q_2 \times dH_2 = 0$$

- Solution

  - Under the assumption of unique *dy* – parallel shifts (impacting simultaneously on the price of the portfolio and the hedging instruments):

$$\begin{cases} P'(y) + q_1 H_1'(y) + q_2 H_2'(y) = 0 \\ P''(y) + q_1 H_1''(y) + q_2 H_2''(y) = 0 \end{cases}$$

→ with non-flat yield curves, the impact of yield changes on bond prices will not result straight from durations or convexities

  - Under the assumption of a unique *y* – flat yield curve => the impact of yield changes on bond prices will result straight from durations and convexities:

$$\begin{cases} q_1 H_1(y) D_1 + q_2 H_2(y) D_2 = -P(y) D_p \\ q_1 H_1(y) C_1 + q_2 H_2(y) C_2 = -P(y) C_p \end{cases}$$

EXAMPLE:

- Portfolio at date t
  - Price P = $ 347000
  - Yield y = 5,13%
  - Duration = 6,78
  - Convexity = 50,26
- Hedging instrument 1
  - Price H_1 = $ 97962
  - Yield y_1 = 5,27%
  - Duration = 8,09
  - Convexity = 73,35
- Hedging instrument 2:
  - Price H_2 = $ 108039
  - Yield y_2 = 4.10%
  - Duration = 2.82
  - Convexity = 8,18

- Optimal quantities q_1 and q_2 of each hedging instrument are given by

$$\begin{cases} q_1 H_1(y) D_1 + q_2 H_2(y) D_2 = -P(y) D_P \\ q_1 H_1(y) C_1 + q_2 H_2(y) C_2 = -P(y) C_P \end{cases}$$

$$\begin{cases} q_1 \cdot 97962 \cdot 8{,}09 + q_2 \cdot 108039 \cdot 2{,}82 = -347000 \cdot 6{,}78 \\ q_1 \cdot 97962 \cdot 73{,}35 + q_2 \cdot 108039 \cdot 8{,}18 = -347000 \cdot 50{,}26 \end{cases}$$

- Solving in order to $q1$ and $q2$:

$$\begin{cases} q_1 = -2{,}17 \\ q_2 = -2{,}07 \end{cases}$$

The investor should sell 2 units of each hedging instrument if the yield curve is (close to) flat, namely be selling 2 futures contracts with the underlying assets being bonds with the corresponding features.

DURATION + CONVEXITY HEDGING - LIMITATIONS

- It is not easy to find hedging contracts with the required features => higher costs to find tailor-made hedging instruments or any hedging will be just an approximation, even assuming that the yield curve is flat.

- Furthermore, the yield curve is not flat => more complex calculations.

- Additionally, yield curve shifts are not only parallel, but its shape also changes => even more complex calculations.

The yield curve dynamics is not fully explained by one factor models => multifactor models are needed.

4 –IR DERIVATIVES

- Forward Rate Agreements (**FRAs**)
- Interest Rate futures
- Interest Rate swaps (**IRS**)
- Interest Rate **Options:**
  - Plain vanilla Bond or short-term futures **Options**
  - Interest rate **CAPS**
  - **Swaptions**

FRAs

- A **forward rate agreement** FRA is a contract involving three time instants: The current time t, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest rate payment for the period $T \mapsto S$ with fixed rate K at maturity S against an interest rate payment over the same period with rate $L(T, S)$.
- Basically, this contract allows one to lock–in the interest rate between T and S at a desired value K.
- *T* is also usually known as the settlement date and the difference between *t* and *T* as the time to settlement.
- If one assumes that investor are risk-neutral, the FRA's interest rate corresponds to the expected interest rate for time *T* and term *S-T*.

FRAs

- FRAs are quoted by financial institutions in over-the-counter market.

- Therefore, they are not listed in exchange markets, even though there is public data on FRAs, from quotes by market participants.

- Additionally, FRAs are quoted for fixed times to settlement and maturities, e.g. 3 x 9 (6-month interest rate forward, with a time to settlement of three months).

- Quotes are in percentage points, as usual interest rates (e.g. 2%, 4%).

- In order to cancel an exposure to a FRA, one cannot sell the contract, but can take a short exposure on a futures contract, for a time to settlement and an interest rate maturity as close as possible.

- Just like for a bond, the value of a FRA contract is an inverse function of the interest rate: an increase in spot rates reduces the value of a FRA contract => for the buyer/holder of a long position in a FRA, interest rate increases are unfavorable.

FRAs

- 3-month FRA with a time to settlement = 3y <-> FRA (3x39)
- As the actual 3-month interest rate at T is higher than the FRA rate, the FRA buyer will lose money:

Example 4.3

Suppose that a company enters into an FRA that specifies it will receive a fixed rate of 4% on a principal of $100 million for a 3-month period starting in 3 years. If 3-month LIBOR proves to be 4.5% for the 3-month period the cash flow to the lender will be

$$100{,}000{,}000 \times (0.04 - 0.045) \times 0.25 = -\$125{,}000$$

at the 3.25-year point. This is equivalent to a cash flow of

$$-\frac{125{,}000}{1 + 0.045 \times 0.25} = -\$123{,}609$$

Assuming simple compounding

at the 3-year point. The cash flow to the party on the opposite side of the transaction will be +$125,000 at the 3.25-year point or +$123,609 at the 3-year point. (All interest rates in this example are expressed with quarterly compounding.)

Source: Hull, John (2009), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 7th Edition

FRAs

• Therefore, there is a positive pay-off to the long (short) position holder when interest rates decrease (increase), being the FRA value at any given time calculated as:

$$V_{\mathrm{FRA}} = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}$$

being:

L = principal amount

R_K = settled FRA rate

R_F = current forward rate for the corresponding time to settlement and maturity

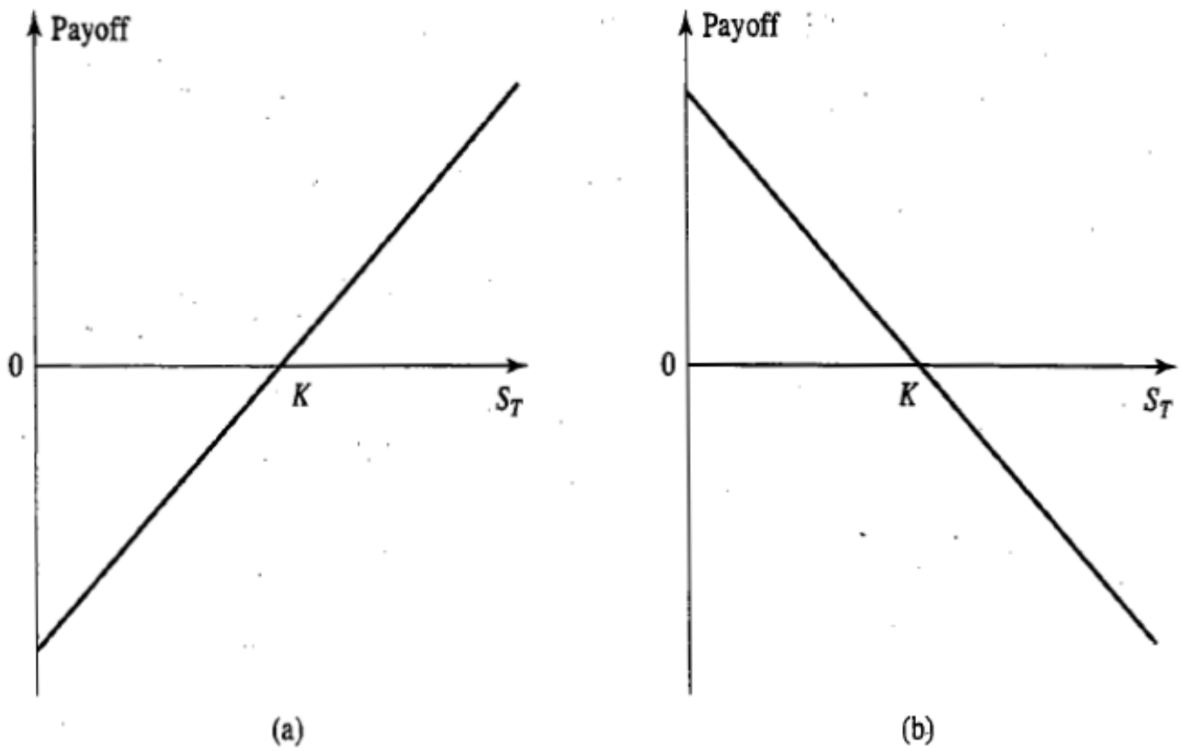
T_1 = maturity date

T_2 = settlement date

•At the negotiation day, the FRA value is zero, **as** the settlement rate is agreed as the market forward rate.

FRAs

Figure 1.2 Payoffs from forward contracts: (a) long position, (b) short position. Delivery price = K; price of asset at contract maturity = S_T.



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INTEREST RATE FUTURES

• Futures contracts are traded in exchange markets, for fixed settlement dates (and consequently different times to settlement and maturity), e.g. the 3-month Euribor futures for Dec.2016 settlement).

•Conversely, FRAs are traded for variable settlement dates and fixed times to settlement and maturity.

• Therefore, in order to cancel a long position in such a contract, an investor can take a short position in (sell) the same contract.

• Usually, one can find futures contracts for short and long term interest rates.

• Short-term futures contracts are quoted as 100 minus the implied interest rates.

• Long-term futures contracts are priced as a % of the nominal value of the theoretical bond embedded in the futures contract, just like the true bonds, being the price of the theoretical bond the NPV of its cash-flows, discounted at current market rates.

Interest Rate Futures

- Therefore, an increase in the price of short-term and long-term futures contracts means that the implied interest rate is decreasing.

- Short-term interest rate futures have financial settlement, while long-term interest rate futures usually have physical settlement, through the cheapest-to-deliver bond (among the bonds considered as deliverable, i.e. proxies for the theoretical underlying bond).

- Short and long term interest rate futures are usually available for quarterly settlement dates (pre-specified days in March, June, September and December).

- However, short term futures are usually available for a longer set of settlement dates (comparing to long term interest rate futures and also FRAs), even though the most liquid contracts are those with shorter times to settlement.

IRS

- IRS are contracts that settle the exchange of fixed for variable interest rates at pre-specified dates.

- Therefore, they may be seen as a long (short) position in a fixed rate bond, a set of FRAs or interest rate futures, on one hand, and a short (long) position in a variable interest rate bond, on the other hand.

- The swap value or price corresponds to its replacement cost, i.e. the amount of money that should be paid by one counterparty to the other to cancel the contract, reflecting the dynamics of short and long term interest rates since the initial date or the last payment date.

- This also corresponds to the difference between a fixed and a floating rate bond:

(from the floating rate payer's point of view) $V_{\text{swap}} = B_{\text{fix}} - B_{\text{fl}}$

- Consequently, at the initial and all payment dates, the swap value returns to 0.

INTEREST RATE OPTIONS

- Interest rate volatilities have to be estimated.
- Consequently, the Dynamics of the yield curve must be assessed => Stochastic Interest Rate Models.
- **Cap** – set of put options on any interest rate to be paid in the future.
- Each of these options is a **caplet** and can be traded individually.
- S**waption** - gives the right to enter into a swap at a future pre-specified rate.
- Contrary to the caps, the several cash-flows of the swaption cannot be traded separately.

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INTEREST RATE OPTIONS

Problem:
We want to price, at t, a European Call, with exercise date S, and strike price K, on an underlying T-bond. $(t < S < T)$.

Naive approach: Use Black-Scholes's formula.

$$F(t,p) = pN\left[d_1\right] - e^{-r(S-t)}KN\left[d_2\right].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{S-t}}\left\{\ln\left(\frac{p}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(S-t)\right\}, \\ d_2 &= d_1 - \sigma\sqrt{S-t}. \end{aligned}$$

where

$$p = p(t,T)$$

Is this allowed?

INTEREST RATE OPTIONS

Difficulties with Black-Scholes pricing:

- The **Dynamics of the yield curve** must be assessed => Stochastic Interest Rate Models.
- **Volatility of the underlying bond varies along time and tends to 0**, while the bond is getting closer to the redemption date.
- **Short-term rates are stochastic** => implementation of dynamic bond pricing models