Exponential smoothing

Based on A. A. Costa,
Notes on pragmatic forecasting procedures and exponential smoothing,
Cemapre 1998

1. Time Series Decomposition

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Usually we consider the two first components jointly as the trend-cycle component

Time Series Decomposition

Given the previous decomposition of time series we can now link all the components using a structural model:

- additive model :

$$y_t = a_t + s_t + e_t$$

- multiplicative model:

$$y_t = a_t \times s_t \times e_t$$

where y_t is the data at period t, a_t is the trend-cycle component, s_t is the seasonal component and e_t is the irregular component.

Trend estimation - Moving Averages

One of the simplest method to filter the erratic component of a series is the use of moving averages.

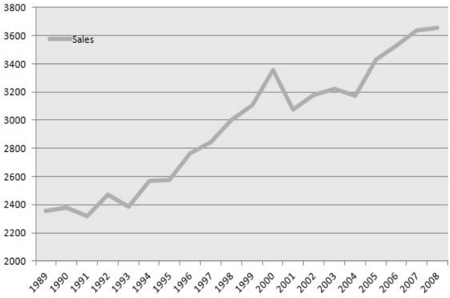
Consider a seasonally adjusted series.

A moving average of odd order m can be written as

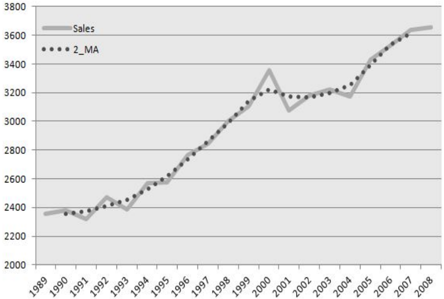
$$M_t = \frac{1}{m} \sum_{j=-k}^{k} y_{t+j},$$

where $m = 2k + 1$ and $t = k + 1, k + 2, ...T - k$.

For a moving average of even order m we have $m = 2k$ and, after centering the series, M_t is defined for $t = m + 1, m + 2, ...$



Sales
3800
3600
3400
3200
3000
2800
2600
2400
2200
2000
1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008



Sales
2_MA
3800
3600
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3000
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2600
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2000
1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008

Seasonality estimation and correction

With seasonal data we can obtain the **seasonality corrected series** using moving averages with period equal/multiple to the number of observations in one year, L, since we can assume that the seasonality component is compensated within a year.

In monthly or quarterly data since L is even we have to recenter the moving averages.

Assuming that the seasonal factor is constant we can obtain the seasonality component and the seasonality corrected series with the method presented in the following slides.

Seasonality estimation and correction

additive model: $y_t = a_t + s_t + e_t$

1. Compute the L-moving averages :

$$M_{t+0.5} = \frac{1}{L} \sum_{i=t+1-\frac{L}{2}}^{t+\frac{L}{2}} y_i \text{ with } t = \tfrac{L}{2}, \tfrac{L}{2}+1, ..., T-\tfrac{L}{2}$$

2. Compute the centered moving averages:

$$M_t = \frac{1}{2} \sum_{i=t-1}^{t} M_{t+0.5} \text{ with } t = \tfrac{L}{2}+1, \tfrac{L}{2}+2, ..., T-\tfrac{L}{2}$$

3. Obtain the seasonal factor for each observation $s_t^* = y_t - M_t$ with $t = \frac{L}{2}+1, \frac{L}{2}+2, ..., T-\frac{L}{2}$
4. Compute the average of the seasonal factors $\bar{s}_i = average(s_{t(i)}^*)$ with $i = 1, 2, ..., L$
5. Normalize the seasonal factors: $\widehat{s}_i = \bar{s}_i - \frac{\sum_i \bar{s}_i}{L}$ for $i = 1, 2, ..., L$
6. Subtract to each observation the corresponding seasonality factor $y_t^D = y_t - \widehat{s}_i(t)$ for $t = 1, 2, ..., T$

Seasonality estimation and correction

multiplicative model: $y_t = a_t \times s_t \times e_t$

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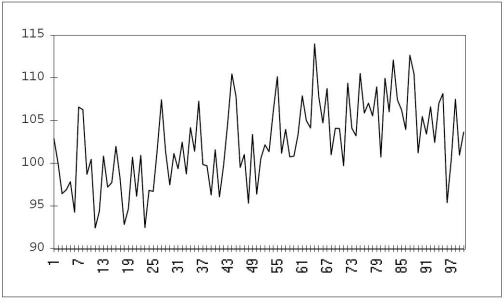
2. Simple exponential smoothing

Non-trend and
locally-constant-level series
$f_{t,h} = \hat{a}_T$ for $h = 1, 2, \ldots$



For forecasting, we should use
an estimate of the local average:

average of all past values?

a moving average?

an average with decaying weights?

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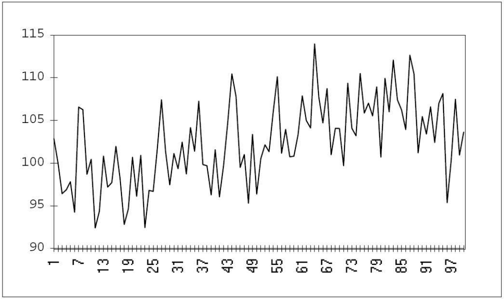

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$$\hat{a}_T = \alpha y_T + \alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \alpha(1-\alpha)^3 y_{T-3} + \ldots$$

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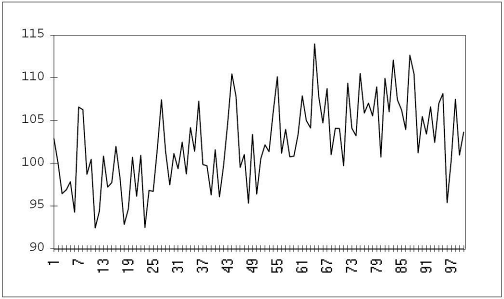

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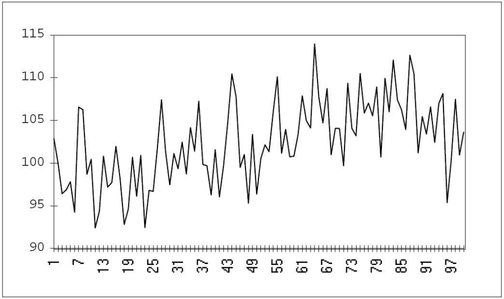

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$$\hat{a}_t = \hat{a}_{t-1} + \alpha(y_t - \hat{a}_{t-1})$$

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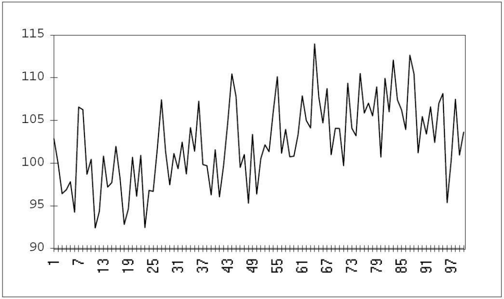

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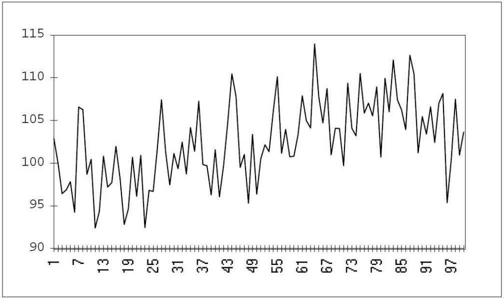

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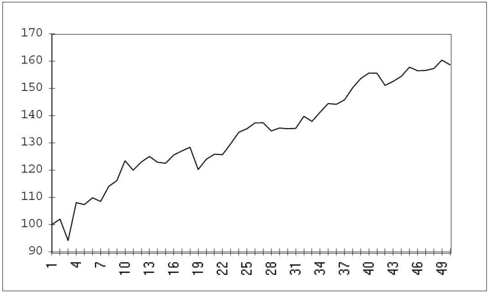
$0 < \alpha < 1 \qquad \alpha \simeq 0.3$

3. Holt's exponential smoothing

Locally-constant trend and locally-constant-level series



3. Holt's exponential smoothing

Locally-constant trend and
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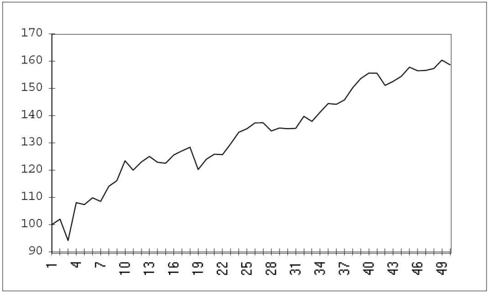
For forecasting, we should use
two estimates of the local average:

local average

local trend



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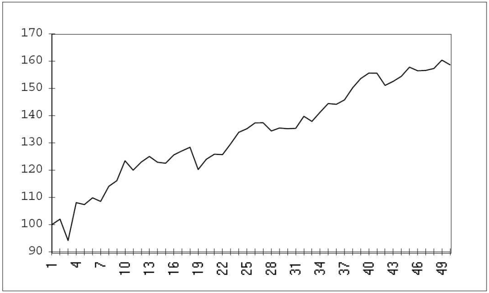

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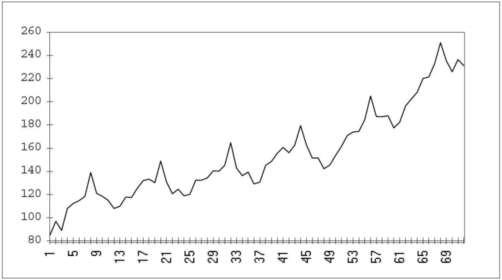
$$0 < \alpha, \beta < 1$$

4. Holt-Winters

"Locally-constant" level, trend, and seasonality



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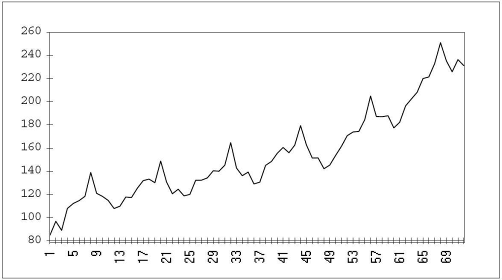
For forecasting, we should use three estimates:

- local level
- local trend
- local seasonality (additive)



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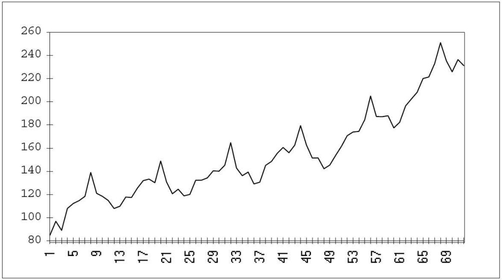

$$f_{t,h} = \hat{a}_t + \hat{b}_t h + \hat{s}_{t+h-kL}$$

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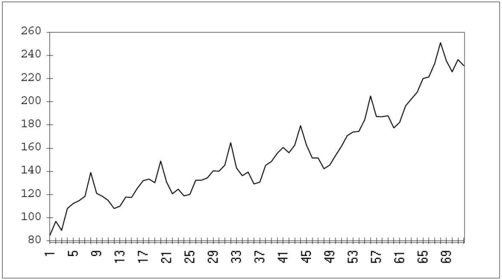

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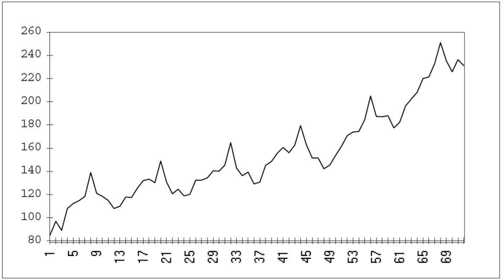

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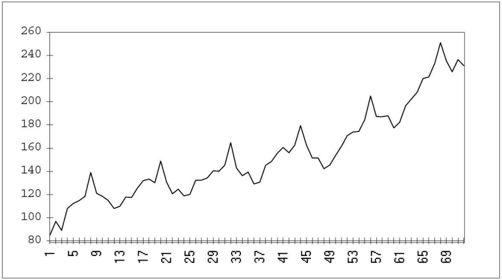

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\end{aligned}
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;

$L =$ "# of months": $k = 1$ if $k \leq L$, $k = 2$ if $L < h \leq 2L$, ...