Problems - Part 1

1. Let S_t be a geometric Brownian motion defined by $S_t = \exp(\mu t + \sigma B_t)$, where B_t is a standard Brownian motion (sBm) and μ and σ are constants.

(a) Write down the SDE satisfied by $X_t = \ln(S_t)$.

(b) By apllying Ito's Lemma (Itô formula), write down the SDE satisfied by S_t.

(c) The price of a share follows a geometric Brownian motion motion with $\mu = 0.06$ and $\sigma = 0.25$ (both expressed in annual units). Find the probability that, over a given 1-year period, the share price will fall.

2. A derivatives trader is modelling the volatility of an equity index using the following time-discrete model (model 1):

$$\sigma_t = 0.12 + 0.4\sigma_{t-1} + 0.05\varepsilon_t, \quad t = 1, 2, 3, \ldots$$

where σ_t is the volatility at time t years and $\varepsilon_1, \varepsilon_2, \ldots$are a sequence of i.i.d. random variables with standard normal distribution. The initial volatility is $\sigma_0 = 0.15$ (that is, 15%). The trader is developing a related continuous-time model for use in derivative pricing. The model is defined by the following SDE (model 2):

$$d\sigma_t = -\alpha(\sigma_t - \mu)\,dt + \beta dB_t,$$

where σ_t is the volatility at time t years, B_t is the standard Brownian motion (sBm) and the parameters α, β and μ all take positive values.

(a) Determine the long-term distribution of σ_t for model 1.

(b) Show that for model 2 (solve the SDE), we have that

$$\sigma_t = \sigma_0 e^{-\alpha t} + \mu\left(1 - e^{-\alpha t}\right) + \int_0^t \beta e^{-\alpha(t-s)} dB_s.$$

(c) Determine the numerical value of μ and a relationship between parameters α and β if it is required that σ_t has the same long-term mean and variance under each model (models 1 and 2)

(d) State another consistency property between the two models that could be used to determine precise numerical values for α and β.

(e) The derivative pricing formula used by the trader involves the squared volatility $V_t = \sigma_t^2$, which represents the variance of the returns on the index. Determine the SDE for V_t in terms of the parameters α, β and μ.