Investments and Portfolio Management

Formulas

Basic Concepts: R_i – random return of asset i ; $\bar{R}_i = \mathbb{E}(R_i)$;

$$\sigma_i^2 = \mathrm{Var}(R_i) = \mathbb{E}[(R_i - \bar{R}_i)^2] = \mathbb{E}(R_i^2) - [\mathbb{E}(R_i)]^2 \; ; \qquad \Rightarrow \qquad \sigma_i = \sqrt{\mathrm{Var}(R_i)} \; ;$$

$$\sigma_{ij} = \mathrm{Cov}(R_i, R_j) = \mathbb{E}[(R_i - \bar{R}_i)(R_j - \bar{R}_j)] \; ; \qquad \Rightarrow \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \; .$$

Portfolios P of n assets: x_i – proportion of the initial wealth W_0 invested in asset i.

$$\bar{R}_P = \sum_{i=1}^{n} x_i \bar{R}_i \qquad \sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} = \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} x_i x_j \sigma_{ij}$$

Homogeneous portfolios:

$$\sigma_H^2 = \frac{1}{n}\overbrace{\left[\sum_{i=1}^{n}\left(\frac{\sigma_i^2}{n}\right)\right]}^{\overline{\sigma_i^2}} + \frac{n-1}{n}\overbrace{\left[\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\frac{\sigma_{ij}}{n(n-1)}\right]}^{\overline{\sigma_{ij}}}$$

Mean-Variance Theory (MVT) $(\, n = 2) \Rightarrow \quad x_1^{MV} = \dfrac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$

MVT – vector notation (n risky assets):

$$\bar{R} = \begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \vdots \\ \bar{R}_n \end{pmatrix} \quad V = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \mathbb{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

- Portfolio ($X_P'\mathbb{1} = 1$): $\bar{R}_P = X_P'\bar{R}$ $\quad \sigma_P^2 = X_P' V X_P$ $\quad \mathrm{Cov}(R_P, R_{P2}) = X_P' V X_{P2}$
- Hyperbola: $$\sigma_P^2 = \frac{A\bar{R}_P^2 - 2B\bar{R}_P + C}{AC - B^2}$$
  where $A = \mathbb{1}'V^{-1}\mathbb{1}$ $\quad B = \mathbb{1}'V^{-1}\bar{R}$ $\quad C = \bar{R}'V^{-1}\bar{R}$.
- Minimum Variance Portfolio: $$X_{MV} = \frac{1}{A}V^{-1}\mathbb{1}$$
- Tangent Portfolio
  - Shortselling allowed: $Z = V^{-1}\left[\bar{R} - R_f\right]$
    - unlimited: $\Rightarrow \quad x_i^T = \dfrac{z_i}{\sum_{i=1}^n z_i} \qquad X_T = \dfrac{Z}{Z'\mathbb{1}}$
    - restricted (Lintner): $\Rightarrow \quad x_i^T = \dfrac{z_i}{\sum_{i=1}^n |z_i|} \qquad X_T = \dfrac{Z}{|Z|'\,\mathbb{1}}$
    - real-life restrictions: $n = 2 \Rightarrow$ trivial , $\; n \geq 3 \Rightarrow$ *numerical solution.*
  - Shortselling forbidden: $n = 2 \Rightarrow$ trivial , $\; n \geq 3 \Rightarrow$ *numerical solution.*

MVT – return generating models:

- Constant Correlation Model: $\rho_{ij} = \rho \quad \forall i, j$

$$\text{Ranking}=\frac{\bar{R}_i - R_f}{\sigma_i} \; ; \; C_k = \frac{\rho \sum_{i=1}^{k} \left(\frac{\bar{R}_i - R_f}{\sigma_i}\right)}{1-\rho+k\rho} \text{(cut-off)} \; ; \; Z_i = \frac{1}{(1-\rho)\sigma_i}\left(\frac{\bar{R}_i - R_f}{\sigma_i} - C^*\right)$$

- Single Index Model: $R_i = \alpha_i + \beta_i R_M + e_i \; ; \quad \sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{e_i}^2 \; ; \quad \sigma_{ij} = \beta_i \beta_j \sigma_M^2$.

Blume's adjust.: $\beta_{2i} = a + b\beta_{1i}$ Vasicek's adjust.: $\beta_{2i} = \frac{\sigma^2_{\beta_{1i}}}{\sigma^2_{\beta_{1i}} + \sigma^2_{\bar{\beta}_1}} \bar{\beta}_1 + \frac{\sigma^2_{\bar{\beta}_1}}{\sigma^2_{\beta_{1i}} + \sigma^2_{\bar{\beta}_1}} \beta_{1i}$

$$\text{Ranking}=\frac{\bar{R}_i - R_f}{\beta_i} \; ; \; C_k = \frac{\sigma_m^2 \sum_{i=1}^{k} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{e_i}^2}}{1 + \sigma_m^2 \sum_{i=1}^{k} \frac{\beta_i^2}{\sigma_{e_i}^2}} \text{(cut-off)} \; ; \; Z_i = \frac{\beta_i}{\sigma_{e_i}^2}\left(\frac{\bar{R}_i - R_f}{\beta_i} - C^*\right)$$

- Multi-index Model: $R_i = a_i + \sum_{k=1}^{K} b_{ik} I_k + c_i; \;\; \sigma_i^2 = \sum_{k=1}^{K} b_{ik}^2 \sigma_{I_k}^2 + \sigma_{c_i}^2; \;\; \sigma_{ij} = \sum_{k=1}^{K} b_{ik} b_{jk} \sigma_{I_k}^2$

Investor Preferences - Expected Utility Theory

- For the utility function $U(W)$

$$\text{ARA: } A(W) = -\frac{U''(W)}{U'(W)} \; ; \qquad \text{RRA: } R(W) = W A(W)$$

2nd ord. Taylor approx. : $U(W) \approx U(W_0) + U'(W_0)(W - W_0) + \frac{1}{2} U''(W_0)(W - W_0)^2$

- Risk tolerance function (RTF) with domain $(\sigma, \bar{R})$ is $f(\sigma, \bar{R}) = \mathbb{E}\left[U(W)\right]$.
- The indifference curves of the RTF are : $f(\sigma, \bar{R}) = K$, for constant levels K.

Investor Preferences - other Geometric average: $\bar{R}_i^G = \prod_{m=1}^{M} (1 + R_{ij})^{p_{ij}} - 1$;

"Safety First": (Roy) $\min \Pr(R_p < R_L)$, (Kataoka) $\max R_L$, *s.a.* $\Pr(R_p < R_L) \le \alpha$ (Telser) $\max \bar{R}_p$, *s.a.* $\Pr(R_p \le R_L) \le \alpha$

Equilibrium Models

CAPM: $\bar{R}_i^{eq} = a + \beta_i b \quad \Rightarrow \quad \bar{R}_i^{eq} = R_f + \beta_i \left(\bar{R}_M - R_f\right) \; ; \quad \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$

APT: $\bar{R}_i^{eq} = \lambda_0 + \sum_{j=1}^{J} b_{ij} \lambda_j \quad \Rightarrow \bar{R}_i^{eq} = R_f + b_{i1}\left(\bar{I}_1 - R_f\right) + \sum_{j=1}^{J} b_{ij}\left(\bar{I}_j - R_f\right)$

Performance indicators:

Sharpe: $SR = \frac{\bar{R}_p - R_f}{\sigma_p}$; Treynor: $TY = \frac{\bar{R}_p - R_f}{\beta_p}$; Jensen: $J = R_p - \left(R_f + \beta_p[\bar{R}_M - R_f]\right)$