

# Financial Markets and Instruments

## Formulas

**Basic Concepts:**  $R_i$  – random return of asset  $i$ ;  $\bar{R}_i = \mathbb{E}(R_i)$ ;

$$\sigma_i^2 = \text{Var}(R_i) = \mathbb{E}[(R_i - \bar{R}_i)^2] = \mathbb{E}(R_i^2) - [\mathbb{E}(R_i)]^2; \Rightarrow \sigma_i = \sqrt{\text{Var}(R_i)};$$

$$\sigma_{ij} = \text{Cov}(R_i, R_j) = \mathbb{E}[(R_i - \bar{R}_i)(R_j - \bar{R}_j)]; \Rightarrow \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}.$$

**Portfolios  $P$  of  $n$  assets:**  $x_i$  – proportion of the initial wealth  $W_0$  invested in asset  $i$ .

$$\bar{R}_P = \sum_{i=1}^n x_i \bar{R}_i \quad \sigma_P^2 = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} = \sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i x_j \sigma_{ij}$$

Homogeneous portfolios:

$$\sigma_H^2 = \frac{1}{n} \left[ \sum_{i=1}^n \overbrace{\left( \frac{\sigma_i^2}{n} \right)}^{\sigma_i^2} \right] + \frac{n-1}{n} \left[ \sum_{i=1}^n \sum_{j=1, j \neq i}^n \overbrace{\frac{\sigma_{ij}}{n(n-1)}}^{\sigma_{ij}} \right]$$

**Mean-Variance Theory (MVT)**

$$(n=2) \Rightarrow x_1^{MV} = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

**MVT – vector notation ( $n$  risky assets):**

$$\bar{R} = \begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \vdots \\ \bar{R}_n \end{pmatrix} \quad V = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

• Portfolio ( $X_P' \mathbf{1} = 1$ ):  $\bar{R}_P = X_P' \bar{R} \quad \sigma_P^2 = X_P' V X_P \quad \text{Cov}(R_P, R_{P2}) = X_P' V X_{P2}$

• Hyperbola: 
$$\sigma_P^2 = \frac{A \bar{R}_P^2 - 2B \bar{R}_P + C}{AC - B^2}$$

where  $A = \mathbf{1}' V^{-1} \mathbf{1} \quad B = \mathbf{1}' V^{-1} \bar{R} \quad C = \bar{R}' V^{-1} \bar{R}.$

• Minimum Variance Portfolio: 
$$X_{MV} = \frac{1}{A} V^{-1} \mathbf{1}$$

• Tangent Portfolio

– Shortselling allowed:

$$Z = V^{-1} [\bar{R} - R_f]$$

\* unlimited: 
$$\Rightarrow x_i^T = \frac{z_i}{\sum_{i=1}^n z_i} \quad X_T = \frac{Z}{Z' \mathbf{1}}$$

\* restricted (Lintner): 
$$\Rightarrow x_i^T = \frac{z_i}{\sum_{i=1}^n |z_i|} \quad X_T = \frac{Z}{|Z|' \mathbf{1}}$$

\* real-life restrictions:  $n=2 \Rightarrow$  trivial,  $n \geq 3 \Rightarrow$  numerical solution.

– Shortselling forbidden:

$n=2 \Rightarrow$  trivial,  $n \geq 3 \Rightarrow$  numerical solution.

## MVT – return generating models:

- Constant Correlation Model:  $\rho_{ij} = \rho \quad \forall i, j$

$$\text{Ranking} = \frac{\bar{R}_i - R_f}{\sigma_i}; C_k = \frac{\rho \sum_{i=1}^k \left( \frac{\bar{R}_i - R_f}{\sigma_i} \right)}{1 - \rho + k\rho} (\text{cut-off}); Z_i = \frac{1}{(1 - \rho)\sigma_i} \left( \frac{\bar{R}_i - R_f}{\sigma_i} - C^* \right)$$

- Single Index Model:  $R_i = \alpha_i + \beta_i R_M + e_i; \quad \sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{e_i}^2; \quad \sigma_{ij} = \beta_i \beta_j \sigma_M^2.$

$$\text{Blume's adjust.: } \beta_{2i} = a + b\beta_{1i} \quad \text{Vasicek's adjust.: } \beta_{2i} = \frac{\sigma_{\beta_{1i}}^2}{\sigma_{\beta_{1i}}^2 + \sigma_{\beta_1}^2} \beta_1 + \frac{\sigma_{\beta_1}^2}{\sigma_{\beta_{1i}}^2 + \sigma_{\beta_1}^2} \beta_{1i}$$

$$\text{Ranking} = \frac{\bar{R}_i - R_f}{\beta_i}; C_k = \frac{\sigma_m^2 \sum_{i=1}^k \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{e_i}^2}}{1 + \sigma_m^2 \sum_{i=1}^k \frac{\beta_i^2}{\sigma_{e_i}^2}} (\text{cut-off}); Z_i = \frac{\beta_i}{\sigma_{e_i}^2} \left( \frac{\bar{R}_i - R_f}{\beta_i} - C^* \right)$$

- Multi-index Model:  $R_i = a_i + \sum_{k=1}^K b_{ik} I_k + c_i; \quad \sigma_i^2 = \sum_{k=1}^K b_{ik}^2 \sigma_{I_k}^2 + \sigma_{c_i}^2; \quad \sigma_{ij} = \sum_{k=1}^K b_{ik} b_{jk} \sigma_{I_k}^2$

## Investor Preferences - Expected Utility Theory

- For the utility function  $U(W)$

$$\text{ARA: } A(W) = -\frac{U''(W)}{U'(W)}; \quad \text{RRA: } R(W) = WA(W)$$

$$\text{2nd ord. Taylor approx. : } U(W) \approx U(W_0) + U'(W_0)(W - W_0) + \frac{1}{2}U''(W_0)(W - W_0)^2$$

- Risk tolerance function (RTF) with domain  $(\sigma, \bar{R})$  is  $f(\sigma, \bar{R}) = \mathbb{E}[U(W)]$ .
- The indifference curves of the RTF are :  $f(\sigma, \bar{R}) = K$ , for constant levels  $K$ .

$$\text{Investor Preferences - other} \quad \text{Geometric average: } \bar{R}_i^G = \prod_{m=1}^M (1 + R_{ij})^{p_{ij}} - 1;$$

$$\text{“Safety First” : (Roy) } \min \Pr(R_p < R_L), \quad \text{(Kataoka) } \max R_L, \quad \text{(Telser) } \max \bar{R}_p, \\ \text{s.a. } \Pr(R_p < R_L) \leq \alpha \quad \text{s.a. } \Pr(R_p \leq R_L) \leq \alpha$$

## Equilibrium Models

$$\text{CAPM: } \bar{R}_i^{eq} = a + \beta_i b \Rightarrow \bar{R}_i^{eq} = R_f + \beta_i (\bar{R}_M - R_f); \quad \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

$$\text{APT: } \bar{R}_i^{eq} = \lambda_0 + \sum_{j=1}^J b_{ij} \lambda_j \Rightarrow \bar{R}_i^{eq} = R_f + b_{i1} (\bar{I}_1 - R_f) + \sum_{j=1}^J b_{ij} (\bar{I}_j - R_f)$$

## Performance indicators:

$$\text{Sharpe: } SR = \frac{\bar{R}_p - R_f}{\sigma_p}; \quad \text{Treynor: } TY = \frac{\bar{R}_p - R_f}{\beta_p}; \quad \text{Jensen: } J = R_p - (R_f + \beta_p [\bar{R}_M - R_f])$$