Financial Markets and Instruments

Booklet of Exercises

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Recommended exercises for the Master level course on *Financial Markets and Instruments*. Students are encouraged to suggest additional exercises and report any typos they may find. The exercises should be solved at home and discussed during study sessions or office hours. Only a very limited number of exercises will be solved during regular classes. All exercise suggestions and/or list of typos should be sent to Rmgaspar@iseg.ulisboa.pt.

Contents

1 Mean–Variance Theory

1.1 Return and Diversification of Risk

Exercise 1.1. Assume that you are considering selecting assets among the following four candidates:

Asset 1		
Market Cond.	Return (%)	Prob.
Good	16	$\frac{1}{4}$
Average	12	$\frac{1}{2}$
Poor	8	$\frac{1}{4}$

Asset 2		
Market Cond.	Return (%)	Prob.
Good	4	$\frac{1}{4}$
Average	6	$\frac{1}{2}$
Poor	8	$\frac{1}{4}$

Asset 3		
Market Cond.	Return (%)	Prob.
Good	20	$\frac{1}{4}$
Average	14	$\frac{1}{2}$
Poor	8	$\frac{1}{4}$

Asset 4		
Rain Fall	Return (%)	Prob.
Plentiful	16	$\frac{1}{3}$
Average	12	$\frac{1}{3}$
Light	8	$\frac{1}{3}$

Assume that there is no relationship between the amount of rainfall and the condition of the stock market.

(a) What are the expected returns and the volatilities of each of the individual assets?

(b) Find out the correlation coefficient and the covariance between each pair of returns. Write down the variance-covariance matrix.

(c) Solve for the expected return and volatility of each of the portfolios shown below:

	Weight of each Asset			
Portfolio	Asset 1	Asset 2	Asset 3	Asset 4
a	$\frac{1}{2}$	$\frac{1}{2}$		
b	$\frac{1}{2}$		$\frac{1}{2}$	
c	$\frac{1}{2}$			$\frac{1}{2}$
d		$\frac{1}{2}$	$\frac{1}{2}$	
e			$\frac{1}{2}$	$\frac{1}{2}$
f	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	
g		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
h	$\frac{1}{3}$		$\frac{1}{3}$	$\frac{1}{3}$
i	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

(d) Plot the basic assets and each of the portfolios in (c) in the volatility / expected return space.

(e) What can you conclude about the efficiency of the 4 assets and their portfolios?

Exercise 1.2. Assume that the average variance of returns for an individual security is 50 and that the average covariance is 10.

(a) What is the expected variance of the homogeneous portfolio of 5, 10, 20, 50 and 100 securities?

(b) How many securities need to be held before the risk of a portfolio is only 10% more than the "minimum risk"?

(c) Is this "minimum risk" attainable? Motivate your answer.

Exercise 1.3. Consider that the diversification ratio is of 60% in Italy and 80% in Belgium.

(a) What is a diversification ratio? Write down its equation and interpret.

(b) If the average variance of a single security is 50, what is the expected variance of a homogeneous portfolio of 5, 20 and 100 securities?

Exercise 1.4. Consider the data in the table below.

Number of Securities	Expected Portfolio Variance
1	46.619
2	26.839
4	16.948
6	13.651
8	12.003
10	11.014
12	10.354
14	9.883
16	9.530
18	9.256
20	9.036
25	8.640
30	8.376
35	8.188
40	8.047
45	7.937
50	7.849
75	7.585
100	7.453
200	7.255
500	7.1337
1000	7.097
∞	7.058

(a) What can you conclude about this market diversification ratio? Explain.

(b) What is the minimum number of securities that an investor who desires a variance lower than 8 should invest in? Why?

1.2 Investment Opportunity Sets and Efficient Frontiers

Exercise 1.5. Securities A and B are combined in a no-shortselling portfolio with zero risk.

(a) What is the weight of each security in the portfolio, assuming the volatility of security A is 9% and the volatility of B is 15%.

(b) Consider that the null risk portfolio has a return of 7.5%. If the expected return of security A is 5%, what can you conclude about the expected return of security B?

(c) Comment the following statement: "*Under this conditions it is never efficient to shortsell security B*".

Exercise 1.6. Consider securities 1 and 2 from Exercise 1.1 and assume shortselling is not possible.

(a) Derive the equations that, in the plan $(\sigma, \bar{R})$, describe all possible combinations of securities 1 and 2.

(b) Find the composition, volatility, and expected return of the minimum variance portfolio.

(c) Assuming that investors prefer more to less and are risk avoiders, indicate those sections of the diagram that are efficient. Justify.

(d) How would you change your answers to (a) – (c) if shortselling would be allowed? Explain.

Exercise 1.7. Consider the following two assets:

Asset 1		
Market Cond.	Return	Prob.
Good	20%	$\frac{1}{3}$
Average	14%	$\frac{1}{3}$
Poor	8%	$\frac{1}{3}$

Asset 2		
Market Cond.	Return	Prob.
Good	16%	$\frac{1}{3}$
Average	12%	$\frac{1}{3}$
Poor	8%	$\frac{1}{3}$

Furthermore, when answering the questions below always consider three alternative scenarios: (i) shortselling allowed, (ii) shortselling limited *a la* Lintner, and (iii) shortselling forbidden.

(a) Represent analytical and graphically for all possible combinations of securities 1 and 2.

(b) What is the composition, volatility and expected return of the minimum variance portfolio.

(c) Assuming that investors prefer more to less and are risk avoiders, indicate those sections of the diagrams that are efficient. Justify.

(d) Suppose now there is also a riskless asset with $R_f = 10\%$.

(i) How would the investment opportunity set change? Sketch it in the mean-variance plan and derive its formulas.

(ii) What would then be the efficient frontier? Explain.

Exercise 1.8. Derive the expression for the location of all portfolios of two securities in expected return standard deviation space when the correlation between the returns of two securities is:

(a) $+1$; (b) -1; (c) 0; (d) any value ρ with $-1 < \rho < 1$ and $\rho \neq 0$.

(e) Discuss how realistic the correlations in (a) – (d) are.

Exercise 1.9. Consider the following two securities

	Expected Return	Standard Deviation
Security 1	10%	5%
Security 2	4%	2%

(a) For the 2 securities shown plot all combinations of the 2 securities in $(\sigma_p, \bar{R}_p)$ space. Assume $\rho = 1, -1, 0$.

(b) For each correlation coefficient and assuming no shortselling what is minimum risk portfolio and its volatility?

(c) Assume now shortselling is allowed and consider again all mentioned correlations. What can you conclude:

(i) About the minimum risk portfolio?

(ii) About the existence or not of a risk-free asset and about R_f when it exists?

Exercise 1.10. In Exercise 1.9, assume a riskless rate of 10%. What is(are) the efficient investment(s)? Why?

Exercise 1.11. Consider the following information on three risky assets, and R_f values for both lending and borrowing rate of 6%, 8%, 10%.

Security	Mean Return (%)	Standard Deviation (%)	Covariance		
			A	B	C
A	11	2		10	4
B	14	6			30
C	17	9			

(a) Derive the equation of the efficient frontier.

(b) What would be the efficient frontier if there would be no credit to invest in risky assets? Explain step by step and write down the appropriate equations.

(c) What would change if shortselling is forbidden? Sketch and solve.

(d) What if shortselling is possible but it is limited?

  (i) Consider shortselling constrained *a la* Lintner. Explain the Lintner definition of portfolio and its connection to shortselling restrictions.

  (ii) Consider there is an upper bound of at most 50% of the portfolio value.

Exercise 1.12. Assume that the data below apply to two *efficient* portfolios.

Portfolio	$\bar{R}_i$	σ_i	
A	10%	6%	
B	8%	4%	
			$\rho_{ij} = +\frac{5}{6}$

(a) What is the efficient frontier? Derive it analytically and sketch it in the mean-variance plan.

(b) Suppose now there is a riskless asset with $R_f = 2\%$, valid for both deposit and borrowing. What would then be the efficient frontier?

(c) How would you change your answers to (a)-(b) if short-selling if forbidden or limited *a la* Lintner? Explain and consider both:

  (i) the case when the portfolios A and B are still feasible.

  (ii) the case when at least one of the portfolios, A or B, is no longer feasible.

1.3 Portfolio Protection

Exercise 1.13. Assume the original setting of Exercise 1.11 with $R_F = 8\%$ and that, in addition that the returns of A, B and C follow a Normal distribution. Suppose you are worried about portfolio protection.

(a) Find out the portfolio that minimizes the likelihood of returns below:

(i) 6% (ii) 8% (iii) 10%

Compare the results and explain.

(b) Find out the portfolio with the highest return-at-risk (RaR) for an $\alpha = 10\%$. Interpret.

(c) Characterize the set of all portfolios that have less than 10% probability of returns lower than 10%.

(d) Classify the safety criteria used in (a)-(c) and discuss their differences.

Exercise 1.14. Consider A and B from Exercise 1.12 and assume their returns are Gaussian.

(a) Identify all combinations of A and B whose probability of returns lower than 15% is less than 5%.

(b) Which combination maximizes the likelihood of getting returns above 5%.

(c) What is combination with the highest 15% quantile of the returns distribution?

(d) Establish the necessary connections between the combinations determined in (a)-(c) and the safety criteria of Roy, Kataoka and Telser. Explain.

(e) If the returns of A and B where not Gaussian, how would you answer the above questions?

Exercise 1.15. Consider two Gaussian risky assets with $\bar{R}_1 = 12\%, \bar{R}_2 = 6\%, \sigma_1 = 20\%, \sigma_2 = 15\%$ and $\rho = +0.5$. Shortselling is allowed without bound but it is not possible to get a loan to invest in risky assets. Still, there exist a riskless rate $R_f = 3\%$ for deposits.

(a) Sketch the investment opportunity set and the efficient frontier in the mean-variance plan. Explain.

(b) Find out the minimum variance portfolio, its expected return and volatility.

(c) Derive the equations of the efficient frontier.

(d) Check if the homogeneous portfolio, H, is efficient. Motivate.

(e) Consider now a portfolio I with $\bar{R}_I = 8\%$ and $\sigma_I = 18\%$.

  (i) What can you conclude about the efficiency of I? Explain.

  (ii) Find out the composition of portfolio I. Motivate all steps.

  (iii) Compare the Sharpe ratio of I with that of the tangent portfolio T.

  (iv) What is the probability that portfolio I has negative returns?

  (v) What is the investment that minimizes the probability of negative returns?

(f) Assume that you are not allowed to consider portfolios that have a probability of negative returns higher than 10%. What can you conclude about:

  (i) The new investment opportunity set.

  (ii) The new efficient frontier.

1.4 International Diversification

Exercise 1.16. Explain what diversification is. Discuss the effect of international diversification in any given portfolio.
(Obs: Consider N assets).

Exercise 1.17. Consider the following returns:

Period	EUA	UK	Exchange Rate* ($/£)
1	10%	5%	3
2	15%	-5%	2.5
3	-5%	15%	2.5
4	12%	8%	2.0
5	6%	10%	1.5
6			2.5

* Beginning of period dollars for pounds

(a) What is the average return in each market from the point of view of a U.S. investor? What about the case of a U.K. investor?

(b) What is the standard deviation of returns from the point of view of a U.S. investor? What about the case of a U.K. investor?

Exercise 1.18. Assume that an US investor expects the average return on a security in various markets is as shown in the following table.

	Market	Expected Return (%)
1.	Austria	14
2.	France	16
3.	Japan	14
4.	United Kingdom	15
5.	United States	20

Assume further that the historical correlation coefficients shown below are a reasonable estimate of future correlation coefficients.

ρ	1.	2.	3.	4.	5.
1.		0.400	0.430	0.543	0.505
2.			0.415	0.642	0.534
3.				0.474	0.348
4.					0.646
5.					

Finally, assume the standard deviations shown next.

	Domestic Risk(%)	Exchange Risk (%)	Total Risk (%)
1.	24.80	10.59	24.50
2.	18.87	10.61	17.76
3.	22.04	12.46	25.70
4.	14.45	10.10	15.59
5.	13.59		13.59

Which markets are attractive investments for an American investor if the riskless lending and borrowing rate is 6%.

Exercise 1.19. Consider the following data on volatilities associated with various domestic D and non-domestic N investments (from the domestic investor point of view).

	Equities(%)	Bonds (%)	T- bills (%)
N	19.00	12.875	10.057
D	15.39	6.916	1.068

What is the minimum risk combination of domestic D ,and non-domestic N, assets:

(a) Equities if $\rho_{N,D} = 0.423$.

(b) Bonds if $\rho_{N,D} = 0.527$.

(c) T-bills if $\rho_{N,D} = -0.220$

2 Return Generating Models

2.1 Constant Correlation Model

Exercise 2.1. Consider Constant Correlation Models (CCMs).

(a) What are the assumption(s) underlying a CCM?

(b) What are the advantages of using a CCM?

Exercise 2.2. Consider the following data.

Portfolio	$\bar{R}_i$	σ_i
1	10	5
2	8	6
3	12	4
4	14	7
5	6	2
6	9	3
7	5	1
8	8	4
9	10	4
10	12	2
	$\rho_{ij} = 0.5\ \forall i, j$	
	$R_f = 4\%$	

(a) Is it appropriate to use a CCM in this setting? Why or why not.

(b) What is the tangent portfolio, T^A, assuming short sales are allowed?

(c) What is the tangent portfolio, T^F, assuming short sales are forbidden?

(d) What is the tangent portfolio, T^L, assuming short sales are limited *a la* Lintner?

(e) Trace out in the mean-variance plan the efficient frontiers in (b)-(d).

Exercise 2.3. Given the following data

Portfolio	$\bar{R}_i$	σ_i
1	15	10
2	20	15
3	18	20
4	12	10
5	10	5
6	14	10
7	16	20

In this economy, the average correlation is 0.5 and the riskless rate is 5%.

(a) Trace out the efficient frontier in the risk/expected return space.

(b) What is the efficient frontier of risky assets?

  (i) When short-selling is not allowed.

  (ii) When short-selling is allowed, using standard definition.

  (iii) When short-selling is allowed, but using Lintner definition.

(c) How would you find all efficient portfolios, if the risk free asset does not exist in this economy? Fundament your answer.

2.2 Single-Index Model

Exercise 2.4. Consider the following data about a given one-factor model, where the factor is a market index M:

Security	Beta	Specific Variance
A	0.75	0.02
B	2	0.03

The volatility of the market index is $\sigma_M = 25\%$.

(a) How would you classify A and B in terms of aggressiveness level? Justify.

(b) Consider a portfolio with 25% of security A and 75% of security B.

  (i) What is the beta of that portfolio?

  (ii) Decompose the variance of that portfolio into diversifiable and non-diversifiable variance. Explain.

(c) Find the composition and beta of a portfolio formed with securities A and B, which risk equals the market risk.

(d) Assume that total variance of securities A and B is 0.1 and 0.3, respectively. Check if the single index model is consistent with this new data.

Exercise 2.5. Consider the following data about two securities:

Security	β_i	σ_{ei}^2
A	0.875	0.10
B	1.125	0.15

The market index has a volatility of 40% and the covariance between the specific components of the returns of A and B is 0.1.

(a) Compute and interpret the covariance between security B's and the market portfolio's return.

(b) Find the residual variance of a homogeneous portfolio with security A and B, considering the Single Index Model assumptions.

(c) Find the effective residual variance of the homogeneous portfolio.

(d) Determine the homogeneous portfolio's systematic risk.

(e) Comment the following statements:

  (i) "The total risk of securities A and B is the same under the Markovitz model and the single index model"

  (ii) "The total risk of a portfolio composed by securities A and B is the same under the Markovitz model and the single index model".

Exercise 2.6. Given the data below and the fact that $\bar{R}_M = 8\%$ and $\sigma_M = 5\%$, calculate the following:

	A	B	C	D
α	2%	3%	1%	4%
β	1.5	1.3	0.8	0.9
σ_{ei}	3%	1%	2%	4%

(a) What model apply to this exercise? Explain its assumptions.

(b) Calculate

  i. The expected return for each security.

  ii. The volatility of each security.

  iii. The covariance of returns between each pair of securities.

(c) Assuming an equally weighted portfolio, H, calculate the following:

  (i) β_H (ii) α_H (iii) σ_H^2 (iv) $\bar{R}_H$

(d) Use Blume's technique where $\beta_{2i} = 0.343 + 0.677\beta_{1i}$ to calculate β_{2i} for each security.

(e) Suppose $\bar{\beta}_1 = 1$, $\sigma_{\bar{\beta}_1} = 0.25$, $\sigma_{\beta_A} = 0.21$, $\sigma_{\beta_B} = 0.32$, $\sigma_{\beta_C} = 0.18$, $\sigma_{\beta_D} = 0.20$, forecast each security's beta using the Vasiček adjustment.

Exercise 2.7. Assume that all assumptions of the single index model hold, except that the covariance between residuals is a constant K instead of zero.

(a) Derive the covariance between the two securities.

(b) Derive the expression valid for the variance of a portfolio.

Exercise 2.8. The market index variance is $\sigma_M^2 = 0.001$ and a single index model can explain the returns of 6 securities.

Security	$\bar{R}_i(\%)$	β_i	σ_{ei}^2
1	15	1.0	0.003
2	12	1.5	0.002
3	11	2.0	0.004
4	8	0.8	0.001
5	9	1.0	0.002
6	14	1.5	0.001

(a) Find the efficient portfolio of risky assets in a scenario where the riskless rate is $R_f = 5\%$ and short sales not allowed.

(b) Consider the same riskless rate of $R_f = 5\%$, what would change if:

  (i) Shortselling is allowed without restrictions?

  (ii) Shortselling is limited *a la* Lintner?

  Motivate our answers.

(c) Assume that short selling is allowed, but there is no riskless security (i.e. it is not possible to borrow or lend at the riskless rate R_f). Explain how you would determine the efficient frontier.

Exercise 2.9. Suppose that, in a given economy, the correlation coefficient between all securities' returns is the same, ρ^*, and that the assumptions of the single index model are valid. Derive an expression for the beta of any security.

Exercise 2.10. Consider the following data:

Securities	α_i	β_i	σ_{ei}	Other data	
A	2%	1,5	4%	$\bar{R}_m$	20%
B	4%	0,8	3%	σ_m	10%
C	6%	0,4	2%	R_F	2%

Using the Market Model, fill in the following table (in % and using two decimal places, e.g. 7.25%):

Securities	Expected Return	Systematic Risk	Specific Risk	Total Risk
A				
B				
C				
Portfolio K				

Portfolio K is composed by 20% of A, 30% of B and the remaining of C.

2.3 Multi-Index Model

Exercise 2.11. Show how to reduce a general three-factor model with possibly correlated factors into a three-factor model with orthogonal factors.

Exercise 2.12. Given a three index model such that all indexes are orthogonal, *derive* the formulas for:

(a) The expected return.

(b) Variance of any stock.

(c) Covariance between two any stocks.

Exercise 2.13. A multi-factor model was estimated for two risky assets:

$$\begin{aligned} R_{A\,t} &= 2\% + 1.1 I^*_{1\,t} + 1.2 I^*_{2\,t} + c_{A\,t} \\ R_{B\,t} &= 5\% + 0.8 I^*_{1\,t} + I^*_{2\,t} + c_{B\,t} \end{aligned}$$

where I^*_1 and I^*_2 are correlated factors that verify the regression $I^*_{2\,t} = 2\% + 1.5 I^*_{1\,t} + d_t$ and we have $\bar{I}^*_1 = 12\%$, $\sigma_{I^*_1} = 20\%$,$\sigma_d = 3\%$. Furthermore we know $\sigma_{c_A} = 5\%, \sigma_{c_B} = 7\%$.

(a) Write the equivalent multi-factor equations for orthogonal factors I_1, I_2.

(b) Determine the mean-variance theory (MVT) inputs.

(c) Assume there is additionally a riskless asset with a deposit rate of $R^p_f = 3\%$ and a credit rate of $R^a_f = 5\%$

  (i) Sketch the investment opportunity set.

  (ii) Write down the equations that describe the efficient frontier. Motivate.

Exercise 2.14. What are the economic variables more often used as explanatory variables in multi index models?

Exercise 2.15. Assume that the following two index model holds:

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + e_i$$

You have the following data on the two uncorrelated indexes, I_1 e I_2: $\bar{R}_{I_1} = 15\%$; $\sigma_{I_1} = 25\%$ and $\sigma_{I_2} = 5\%$. You also have information on 3 securities (see table below) :

Security	Expected Return (%)	a_i (%)	b_{i1}	b_{i2}	σ_{ei} (%)
A	——	0.2	1.2	-0.15	5
B	12.5	0.5	0.8	0	2
C	9.5	-0.1	0	1.2	1

Finally, in this economy exists a riskless asset with return rate $R_f = 5\%$, that can be used for both lending and borrowing.

(a) What are the "b's" associated to the riskless asset?

(b) What is the expected value for index 2?

(c) What is the expected value for security A ?

(d) What is the systematic risk of each security A, B and C?

(e) What is the covariance matrix for the 3 securities?

(f) Assume that you can only invest in securities B and C.

  (i) Determine the composition and the risk of the minimum variance portfolio.

  (ii) If you could invest in the riskless asset, what would be the minimum variance portfolio?

(g) Assume hat you can invest in any security, A, B and C and that short-selling is allowed.

  (i) Find the composition of the efficient portfolio of risky assets, portfolio T, and its expected return and risk.

  (ii) Write the expression for the efficient frontier.

3 Selecting the Optimal Porfolio

3.1 Expected Utility Theory

Exercise 3.1. Show that when presented with a fair game:

(a) a risk-neutral agent is indifferent between entering the game or not

(b) a risk-averse agent does not enter the game.

(c) a risk-lover agent always chooses the game

Exercise 3.2. Suppose our investor has power utility $U(W) = -W^{-1/3}$, and we wish to choose between investments A and B which result in final total wealths as follows:

A		B	
probabilities	value	probabilities	value
0.25	4	0.333333	4
0.5	6	0.333333	6.2
0.25	8	0.333333	8

(a) What investment do you recommend? Explain.

(b) What if we have power utility but with a different power $U(W) = -W^{-0.1}$.

(c) Compare the two investors.

Exercise 3.3. Suppose a coin is tossed twice. On heads wealth is doubled. On tails wealth is multiplied by 0.05. The investor has log utility. Initial wealth is 1 000. How much wealth would the investor pay to avoid this situation?

Exercise 3.4. Consider two investors, one with exponential utility $U(W) = 1 - e^{-0.001W}$ and another with log utility $U(W) = \ln(W)$. To both investors was proposed the following far game: to win X to loose X with equal probability.

(a) Suppose both investors have an initial wealth of €1 000.

   (i) How much would they pay to avoid the game? Why?

   (ii) Is there an optimal X value?

(b) What if their initial wealth is €100 000? Interpret your results.

Exercise 3.5. Consider an investor with utility function $U(W) = W - 0.005W^2$ with initial wealth $W_0 = 50$. He was offered the possibility of entering a fair game, where he could loose or win 25 with equal probability.

(a) Is he willing to play the game? Why or why not?

(b) How could you distort the game probabilities so that he would be indifference between playing or not?

(c) What is his certainty equivalent to the game? How to interpret it?

Exercise 3.6. Suggest an utility function that is able to describe preferences of an investor with initial wealth €2 500, that ranks three risk project as $X \succ Y \succ Z$. About the risky projects we only know $\bar{R}_X = 20\%$, $\bar{R}_Y = 15\%$ and $\bar{R}_Z = 8\%$, while $\sigma_X = 30\%$, $\sigma_Y = 35\%$, $\sigma_Z = 5\%$. Explain. Study the suggested utility in terms of absolute and relative risk aversion.

Exercise 3.7. Consider the following *alternative* investments.

Invest. A		Invest. B		Invest. C	
Outcome	Prob.	Outcome	Prob.	Outcome	Prob.
5	1/3	4	1/4	1	1/5
6	1/3	7	1/2	9	3/5
9	1/3	10	1/4	18	1/5

(a) Assuming the utility function of a given investor 1 is:

$$U(W) = 20W - 0.5W^2 \tag{1}$$

What is the preferred investment?

(b) Imagine a second investor whose utility function is

$$U(W) = -\frac{1}{\sqrt{W}} \tag{2}$$

Would this second investor agree with the first investor choices?

(c) Consider investment B. The probability associated to outcomes 4 and 10 is 1/4. What should the probability of these equally likely outcomes, so that investor 1 is indifferent between investments A and B?

(d) Consider now investments B and C. To what value you have to change the lowest outcome of investment C to make these two investments indifferent to investor 2?

Exercise 3.8. Consider again the utility function (2).

(a) What can be said about the investor behaviour towards risk?

(b) Study its characteristics and interpret in terms of absolute and relative aversion to risk.

Exercise 3.9. Suppose the utility function of a given investor is $U(W) = ae^{-bW}$, with a and b constants.

(a) What sign must a and b assume in order to make this investor risk averse and to prefer more to less?

(b) Suppose this investor has the chance to do a risky investment that requires an initial amount of 1 000 euros. If he decides to invest, it is equally likely that he wins 500 euros or looses 300.

  (i) Show that his decision to invest or not does not depend on the parameter a.

  (ii) Determine the certainty equivalent for the proposed investment as a function of b. Interpret.

  (iii) For $b = 0.01$, compute the certainty equivalent, the risk premium and state the investor choice (in terns of investing or not).

Exercise 3.10. Wealth's W utility function for a certain investor is

$$U(W) = 50W - \frac{1}{2}W^2 \ .$$

Now consider the following investment projects:

X		Y	
€ Outcome	Probability	€ Outcome	Probability
10	0.1	20	0.05
40	0.2	40	0.9
25	0.7	45	0.05

(a) Represent graphically the utility function.

(b) Characterize the investor behavior towards risk.

(c) What project will be chosen by the investor? Explain.

(d) What is the risk premium associated to each project?

Exercise 3.11. An investor, whose risk profile can be expressed by an utility function, wishes to invest in financial markets. To build her utility function, we asked her to give an utility value to some return rates.

Return Rates	*Utility Level*
0 %	0
10%	5

Moreover, she has considered three risky projects X, Y, Z with the following return distributions:

X		Y		Z	
Return	Prob.	Return	Prob.	Return	Prob.
0%	0.5	10%	0.4	-10%	0.2
30%	0.5	30%	0.6	20%	0.8

and we were able to find each project certain equivalents for the projects $C_X = 10\%$, $C_Y = 20\%$ and $C_Z = 10\%$.

(a) Implicitly derive the utility levels associated with returns of $10\%, 20\%$ and 30%. What can you conclude about the investor's attitude towards risk?

(b) Compute the risk premia associated with projects X, Y, Z. Explain.

(c) What are the theoretical foundations of this analysis?

(d) Rank the three projects according to the maximal expected utility principle.

(e) For each of the projects find out the equivalent projects that pay 0 or $b = 30\%$ with the subjective probabilities. Use these distorted probabilities to confirm the ranking in (d).

Exercise 3.12. Consider the following utility function,

$$U(W) = 2 + 4 \ln W$$

(a) Find the absolute and relative risk averse coefficient and explain the meaning of your results.

(b) Consider the following data on the expected wealth of three investment projects X, Y, Z

$$\mathbb{E}[W_X] = 100 \qquad \mathbb{E}[W_Y] = 60 \qquad \mathbb{E}[W_Z] = 70 \ .$$

and suppose each project has just two possible outcomes: 0 or 200.

(i) Determine the implicit real probabilities associated with each outcome for each project.

(ii) Rank the three projects according to the expected utility criteria using the function $U(.)$ above.

(iii) Determine the certainty equivalent and risk premia of each project for an investor with $U(.)$. Interpret.

(c) Consider now the utility function $V(W) = 8 \ln W$. Would your answers to the questions in (b) change? Why or why not?

Exercise 3.13. A given investor has the following utility function

$$U(W) = W - 6W^2 \qquad \text{with} \qquad W < \frac{1}{12}$$

(a) Classify this investor in terms of her risk profile and plot in the risk/expected return space, the shape of her indifference curves. Justify.

(b) Find her absolute and relative risk aversion coefficients.

(c) Explain the differences between absolute and relative risk aversion.

Exercise 3.14. An investor risk tolerance is well captured by the following family of indifference curves $\bar{R}_p = \exp(0.7\sigma_p) + K$, for K constant.

(a) Explain the connection between the utility function, $U(W)$, the risk tolerance function, $f(\sigma_p, \bar{R}_p)$ and its indifference curves.

(b) Characterize the risk profile of the above mentioned investor based upon the shape of the indifference curves.

(c) In a market where the efficient frontier is given by $\bar{R}_p = 0.05 + 0.8\sigma_p$, what is the optimal risk level for our investor? Explain.

Exercise 3.15. Consider a quadratic utility of the form $U(W) = aW^2 + bW + c$.

(a) Under which conditions can the above utility function be used to model (i) risk averse, (ii) risk loving or (iii) risk neutral investors.

(b) Derive the risk tolerance function (RTF) associated with quadratic utility.

(c) Find out the indifference curves of the RTF in (b) and check their slope. Establish a connection to your conclusions in (a).

(d) Consider the case of a risk averse investor and study the absolute and relative risk aversion properties implied by the quadratic utility.

(e) Why are quadratic utility functions so much used in finance?

Exercise 3.16. Consider the data about the following securities:

Securities	$\bar{R}_i$	σ_i
A	8%	10%
B	12%	20%
C	15%	25%
F	4%	0%

You also known that security A is *independent* from the others and the correlation coefficient between B and C is -0.6.

Calculate:

(a) The composition and return of a portfolio formed by securities A and B and a risk level of 9.22%.

(b) The composition and risk of a portfolio with securities A and B and a return level 11%

(c) What is the only efficient combination of just risky assets? Explain.

(d) Write the equation of the efficient frontier for this market?

(e) An investor wants to invest 400 000 euros. If her indifference curves in the plan $(\sigma_p, \bar{R}_p)$ are given by $\bar{R}_p = 0.5\sigma_p^2 + 0.965\sigma_p + 0.01K\ \forall K$.

(i) Based upon the shape of the indifference curves what can you conclude about the investor's attitude towards risk?

(ii) What is the optimal investment, O, for her? What return should she expect?

(iii) Compare the value of her risk tolerance function for the investments T (tangent portfolio), O (optimal portfolio) and F (100% in deposit). Interpret.

(f) Consider now a second investor that is a Log-investor, $U(W) = \ln(W)$.

(i) Explain the problem that arises when one tries to obtain the risk tolerance function (RTF) of the second investor.

(ii) Use a second order Taylor approximation of the utility function to obtain an approximation of his RTF.

(iii) Use the RTF approximation from (ii) to derive the optimal investment.

(iv) Derive the indifference curve of the approximated RTF.

(v) Use now the indifference curves to obtain the optimal portfolio and check you get the same portfolio as in (iii). Explain.

(g) What would be the optimal investment for any investor who is risk neutral? Formalize you answer.

(h) What would be the optimal investment for any investor who is risk lover? Formalize you answer.

3.2 Alternatives Techniques

Exercise 3.17. Consider the three investments presented in Exercise 3.7. Using geometric mean return as a criterion and an initial amount of 100 euros:

(a) Rank the three investments.

(b) What is the rational of using geometric mean as a decision criterion for financial investments?

(c) Check that we get the exact same ordering if we use the expected utility principle with $U(W) = \ln(W)$.

(d) Show that is always the case. For simplicity you can consider only investments with discrete distribution of outcomes.

Exercise 3.18. Given the following *alternative* investments.

A		B		C	
Prob.	Return	Prob.	Return	Prob.	Payoff
0.2	4%	0.1	5%	0.4	6%
0.3	6%	0.3	6%	0.3	7%
0.4	8%	0.2	7%	0.2	8%
0.1	10%	0.3	8%	0.1	10%
		0.1	9%		

(a) What can be said about the investment decision, using the first and second order stochastic dominance criterion?

(b) Suggest an utility function that would lead to the same decision as in (a). Explain your suggestion.

(c) If $R_L = 5\%$, what is the preferred investment using Roy's *safety first* criterion?

(d) If $\alpha = 10\%$, what is the preferred investment using Kataoka's safety-first criterion?

(e) If $R_L = 5\%$ and $\alpha = 10\%$, what is the preferred investment using Telser's safety-first criterion?

(f) Using the geometric mean return as a criterion, which is the preferred investment?

Exercise 3.19. Consider 3 investments A, B and C whose returns follow normal distributions. Furthermore, we know $\bar{R}_A = 10\%$, $\sigma_A = 15\%$, $\bar{R}_B = 12\%$, $\sigma_B = 17\%$,$\bar{R}_C = 15\%$ and $\sigma_C = 30\%$.

(a) What can be said about the investment decision, using the first and second order stochastic dominance criterion?

(b) If $R_L = 5\%$, what is the preferred investment using Roy's *safety first* criterion?

(c) If $\alpha = 10\%$, what is the preferred investment using Kataoka's safety-first criterion?

(d) If $R_L = 5\%$ and $\alpha = 10\%$, what is the preferred investment using Telser's safety-first criterion?

(e) What will be the best investment if the objective is to minimize RaR with $\alpha = 2.5\%$?

4 Equilibrium in Financial Markets

4.1 CAPM

Exercise 4.1. Consider the following data about equilibrium returns in a given capital market:

Stock	$\bar{R}_i^e$	Beta	Specfic Variance
A	0.2	1.5	0.05
B	0.1	0	0.15

Furthermore, assume CAPM holds and the market risk is $\sigma_M = 0.5$.

(a) Find security's A volatility.

(b) Consider the existence of a third security C, with variance 0.75 and without diversifiable risk. Find its beta. What kind of stock is C?

(c) Calculate the market portfolio, M expected return.

(d) What assumptions support the previous result?

Exercise 4.2. Consider the following data about a capital market:

Security	Risk	Expected Return
Market portfolio	10	10%
Treasury Bonds	0	5%

(a) Assume that CAPM holds and the return of an asset A is perfectly and positively correlated with the market return. Determine the expected return for this asset, assuming its risk is 20%.

(b) Calculate e represent geometrically the characteristic line of this asset.

Exercise 4.3. Assume CAPM holds. Consider the following data about the capital market in which security X is listed:

$$\begin{aligned} Cov(R_X, R_M) &= 0.2 \\ \sigma_M^2 &= 0.25 \end{aligned}$$

The riskless return rate is 7%.

(a) If the expected market return is 9%, what is the expected return for asset security X?

(b) If the expected market return is 12% and the expected stock X return is 10%, is the price of this stock in equilibrium? Explain.

Exercise 4.4. Consider the following data:

Stocks	Weights		$\bar{R}_i$	β_i
	Portfolio 1	Portfolio 2		
A	-0.5	0	0.12	1.5
B	0	-1	0.10	1
C	1.5	2	0.05	0.5

Determine the portfolios betas and explain what is the arbitrage strategy that could be implemented here, if any.

Exercise 4.5. Consider the following data about the financial market in which stocks A, B and C traded. These stocks are not alone in this market.

	Security A	Security B	Security C	Portfolio Market
Expected Return in Equilibrium	—	—	2%	10%
Risk	12%	12%	0%	4%
Beta	0.5	-0.1	—	—

(a) Determine the equilibrium conditions and fill the gaps in the table above.

(b) Decompose the risk of stock A and B.

(c) Explain carefully the way investment decisions are made under CAPM assumptions.

Exercise 4.6. Suppose the following assets are correctly priced according to the security market line.

$$\bar{R}_1^e = 6\% \qquad \beta_1 = 0.5$$
$$\bar{R}_2^e = 12\% \qquad \beta_2 = 1.5$$

(a) Derive the security market line.

(b) What is the expected return on an asset with a beta of 2?

(c) Assume that an asset exists with $\bar{R}_3 = 15\%$ and $\beta_3 = 1.2$. Verify there are arbitrage opportunities. If they exist, design a strategy to exploit it.

Exercise 4.7. Assume the security market line:

$$\bar{R}_i = 0.04 + 0.08\beta_i \ .$$

Assume that analysts have estimated the beta on two stocks X and Y as follows: $\beta_X = 0.5$ e $\beta_Y = 2$. What must the expected return on the two securities be in order for them to be a good purchase?

Exercise 4.8. Assume that over some period CAPM equilibrium estimates give

$$\bar{R}_i^e = 0.04 + 0.19\beta_i \ .$$

Assume that over the same period two mutual funds had the following results:

Funds	Actual Return	Beta
A	10%	0.8
B	15%	1.2

What can be said about each fund *performance*?

Exercise 4.9. If the following assets are correctly priced on the security market line:

$$\bar{R}_1 = 9.4\% \qquad \beta_1 = 0.8$$
$$\bar{R}_2 = 13.4\% \qquad \beta_2 = 1.3$$

(a) What is the return of the market portfolio?

(b) What is the risk-free rate?

Exercise 4.10. Assume that returns are generated as follows:

$$R_i = \bar{R}_i + a_i(R_M - \bar{R}_M) + b_i(C - \bar{C})$$

where C is the rate of change in interest rates. Derive a general equilibrium relationship for security returns.

Exercise 4.11. Assume $\bar{R}_M = 15\%$, $R_f = 5\%$ and that risk-free lending is allowed but riskless borrowing is not.

(a) Sketch the efficient frontier in expected return standard deviation space.

(b) Sketch the security market line and the location of all portfolios in expected return Beta space. Label all points and explain why you have drawn them as you have.

Exercise 4.12. Assume you pay a higher tax on income than on capital gains. Furthermore, assume that you believe that prices are determined by the post-tax CAPM. Another investor, however, believes that prices are determined by the pre-tax CAPM. Show that you can make an excess return by engaging in a two-security swap with him.

4.2 APT

Exercise 4.13. Suppose returns are explained by the multi-index model:

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + \epsilon_i \ .$$

Assume a two-factor APT holds, and consider the following information on three portfolios that are known to be in equilibrium:

Portfolio	$\bar{R}_i^e$	b_{i1}	b_{i2}
X	16%	1.0	0.7
Y	14%	0.6	1.0
Z	11%	0.5	-1.5

(a) Find the respective equilibrium relationship.

(b) If a portfolio W, with $\bar{R}_W = 13\%$, $b_{1W} = 1$ and $b_{2W} = 0$ exists, what can we say about existence of arbitrage opportunities? Explain.

(c) Comment the following statement: "*CAPM is a particular case of APT*".

Exercise 4.14. Assume that the multi-index model below correctly explains returns in a given market

$$R_i = a_i + b_{i1} F_1 + b_{i2} F_2 + c_i \ .$$

Consider the following information about three equilibrium returns:

Stocks	$\bar{R}_i^e$	b_{i1}	b_{i2}
A	10%	0.5	1.0
B	12%	1.0	1.5
C	11%	-0.5	2.0

(a) Assume that APT holds.

  (i) Find the equation that explains returns at equilibrium.

  (ii) What is the equilibrium risk-free rate?

(b) Suppose that in this market exists another stock D with: $\bar{R}_D = 12\%$, $b_{D1} = 2$ and $b_{D2} = 0.5$. How would you characterize this stock?

(c) Using stocks A, B and C, find the composition of a portfolio with a risk level identical to stock D. Admit that short-selling is allowed.

Exercise 4.15. The following data about a given capital market is known:

	Security 1	Security 2	Security 3
Expected Return	12%	15%	40%
Expected Return at Equilibrium	10%	20%	—
Returns sensitivity to I_1	1	2	3

Admit that a one-factor APT holds.

(a) Calculate the composition and the return of an arbitrage portfolio.

(b) What is the equilibrium relationship in this model? Fill the gaps in the table above and for each stock made your investment decision.

(c) "*Considering transactions costs in CAPM is consistent with a non-linear relationship between betas and the stock's equilibrium return*". Comment and explain.

Exercise 4.16. Assume that a two-factor APT model holds and consider the following data about three portfolios that are in equilibrium

Portfolio	$\bar{R}_i^e$	b_{i1}	b_{i2}
A	12%	1.0	0.5
B	13.4%	3.0	10.2
C	12%	3.0	-0.5

If $(\bar{R}_M - R_f) = 4\%$, find the values for the following variables that would make the equilibrium expected returns also consistent with Sharpe-Lintner-Mossin CAPM.

(a) $\beta_{\lambda 1}$ and $\beta_{\lambda 2}$.

(b) β_p for each of the three portfolios.

(c) R_f.

Exercise 4.17. In a given market, returns are explained by a two-factor model, with factors I_1 and I_2. Thereafter, we also know:

Portfolio	$\bar{R}_i$	b_{i1}	b_{i2}
X	19%	1.0	0.5
Y	14%	1.4	0
Z	8%	3.0	-1.0

(a) Assume that APT holds and the above portfolios are in equilibrium. Determine:

  (i) the respective equilibrium relationship

  (ii) the expected return for each indexes and

  (iii) the risk-less return rate.

(b) Suppose that in this market you can find a portfolio W with $\mathbb{E}[R_W] = 13\%$, $b_{W1} = 1$ and $b_{W2} = 0$. Could we speak in arbitrage opportunities?

(c) A given fund (F), with an extremely well-diversified portfolio, $b_{1F} = 1.2$ and $b_{2F} = 0.2$, announced a Sharpe Ratio of 0.75. Knowing that the volatilities of the factors I_1 and I_2 are 10% and 25%, respectively, what can you conclude about this fund performance?

(d) A financial analyst believes that CAPM holds in this market with $\bar{R}_M = 15\%$, $\sigma_M = 20\%$. In you opinion, is this possible? Support you idea and, if possible, derive CAPM expected factor values $\bar{I}_1$ and $\bar{I}_2$ and betas $\beta_{I_1}, \beta I_2$, as well as the betas of the X, Y e Z portfolios.

Exercise 4.18. In a given market two financial analysts don't seem to agree. One believe returns are fully explained by a two-indexes model I_1 e I_2 and the other prefer CAPM.

(a) Compare and contrast CAPM and APT.

(b) To the analyst that believes APT is correct, the equilibrium relation ship is $\bar{R}_i = 0.07 + 0.03b_1 + 0.05b_2$. Determine(i) expected returns for the indexes I_1, I_2 and (ii) the risk-less return rate.

(c) On the other hand, CAPM's investor knows that in equilibrium, two portfolios X and Y have: $\bar{R}_X = 30.4\%$, $\beta_X = 1.8$, $\bar{R}_Y = 13.5\%$ and $\beta_Y = 0.5$. Derive, explaining, (i) the equilibrium relationship (ii) market portfolio return (iii) and the risk-less interest rate.

(d) Could both analysts be right? Under what circumstances? Explain. If your answer is positive derive the relationship between(bs) from APT that would apply to portfolios X and Y as well as the betas that would apply to indexes I_1 and I_2.

5 Portfolio Management

Exercise 5.1. Here are data on five mutual funds:

Fund	Return	Volatility	Beta
A	14	6	1.5
B	12	4	0.5
C	16	8	1.0
D	10	6	0.5
E	20	10	2

The risk-free rate is 3% the market return is 13% and its standard deviation is 5%

(a) Is volatility a good risk measure?

(b) What is the reward-to-variability ratio and ranking?

(c) What is the differential return when standard deviation is considered to be a good risk measure?

(d) What is the differential return if Beta is the appropriate measure of risk?

(e) What is the Treynor measure and ranking?

(f) Assume that the zero Beta form of the capital asset pricing model (CAPM) is appropriate. What is the differential return for the funds if $R_z = 4\%$?

(g) For Funds A and B, how much would the return on B have to change to reverse the ranking using the reward-to-variability ratio?

Exercise 5.2. Consider que following data

Funds	Return	Beta	Specific Variance
A	15%	1.3	0.003
B	9%	0.9	0.04

If the risk-free rate is 5% and the market risk is 0.3, what is Sharpe's Ratio of Funds A and B?

6 Miscellaneous

Exercise 6.1. In a famous *country near by the sea* there are two kinds of investors: the *super averse* and the *simply averse*. The preferences of the super averse investors are well described by the following risk tolerance function:

$$f(\bar{R}, \sigma) = 12\bar{R} - \bar{R}^2 - \sigma^2 ,$$

with $\bar{R} = \mathbb{E}(R)$ and $\sigma^2 = \mathrm{Var}(R)$ where R is the return of investments in financial markets.

The next table presents some information on that *country near by the sea*'s capital market:

	Tangent Portfolio T	Risk free asset F	Portfolio A
$\bar{R}_i$	12%	4%	8%
σ_i	6%	0	3%
$\rho_{i,M}$	1	0	1

a. Assume that any investor can deposit or take (unrestricted) loans at the risk free rate.

   (i) What is the set of efficient portfolios of this *country near by the sea*? Check whether portfolio A belongs to that set.

   (ii) Denote by O the optimal portfolio of a *super averse* investor. Show that this optimal portfolio has the following characteristics: $\bar{R}_O = 5.297\%$ and $\sigma_O = 0.96\%$. What is the proportion of the optimal portfolio invested in the risk free asset?

   (iii) Knowing the optimal portfolio of the *simply averse* investors requires investing 120% in the tangent portfolio T, what can you conclude about their optimal risk level?

   (iv) We also know that in this *country near by the sea* there are 1 million of super averse investors and 4 million of simply averse investors. Assume each super averse investor invests 1000 euros in the capital market, while each simply averse investor invests 2000 euros. Check if there is equilibrium.

b. Suppose now taking a loan for investment in financial assets is still possible, but at a higher active interest rate of 7%. In this case we know there is another portfolio B, that involves investing only in risky assets and that is efficient. For portfolio B we also have $\bar{R}_B = 15\%$, $\sigma_B = 12\%$ and we know the correlation between the returns of portfolio T and B is 0.6.

   (i) What is the minimum variance MV portfolio in this case? Justify your derivations.

   (ii) Sketch the efficient frontier when, as in this case, we have active and passive interest rates different from each other. Explain.

   (iii) Assume the optimal risk level of both kinds of investors does not change. Derive their optimal portfolios under the new conditions?

Exercise 6.2. In a country called "Cavaquistão" all efficient portfolios are well described by the expression

$$\bar{R}_p = 3.5\% + 0.3436\sigma_p \ .$$

a. Based upon the expression of the efficient frontier, what can you conclude about (i) the existence (or not) of the risky asset; and (ii) the Sharpe ratio of the tangent portfolio T? Explain your answer

b. We know Mr. Silva preferences are well represented by $U(W) = 50W - 0.01W^2$ and that he wishes to invest 1 000 euros in this market.

   (i) What are the absolute and relative risk aversion coefficients of Mr.Silva before investment? How can you interpret the his absolute and relative risk aversion functions?

   (ii) What is a risk tolerance function? Derive Mr. Silva's.

   (iii) What is the optimal risk level for Mr. Silva?

c. Knowing that the tangent portfolio has only two assets with return distributions as in the following table

Scenarios	Probability	Asset 1	Asset 2
Bad	0.25	-5%	10%
Average	0.5	0%	-5%
Good	0.25	50%	35%

   (i) Determine the composition and risk of the tangent portfolio T?

   (ii) What is the composition and expected return of Mr.Silva's optimal portfolio O?

   (iii) Would the optimal portfolio of Mr. Silva change if the active risk free rate would increase to 10%? Motivate your answer.

Exercise 6.3. The following information concerns three risk assets in a particular market

Assets	$\bar{R}_i(\%)$	$\sigma_i(\%)$
A	8	10
B	12	20
C	15	25

The average correlation across the three assets' returns is +0.5.
In this market it is possible to deposit or borrow money at the same risk-free rate 3%.

a. Assume you are allowed to invest in the three assets and that shortseling in not restricted.

   (i) Find out the portfolio tangent to the risky assets' investment oportunity set (portfolio T).

   (ii) What are the expected return and risk of portfolio T?

   (iii) Derive the efficient frontier equation. Motivate your answer.

b. An investor, whose risk profile is compatible with the indifference cuves $\bar{R}_p = \sigma_p^2 + 0.415\sigma_p + K$ for $K \in \mathbb{R}$, wishes to invest in this market.

  (i) What is the investor's optimal risk level?

  (ii) What are the composition and expected return of his optimal portfolio?

c. In case shortselling would be limited *a la* Lintner, what would change?

  (i) In terms of the shape of the efficient frontier? Explain.

  (ii) In terms of the optimal for our investor? Explain.

d. Looking now at A, B and C as alternative investments and assuming their returns follow a normal distribution, what can you conclude about the ranking of the three assets according to Roy (take $R_L = 5\%$)? Explain.