Probability Theory and Stochastic Processes

LIST 3

Distributions

(1) Determine the distribution functions of the following probability measures on $\mathbb{R}$ (δ_a is the Dirac measure at a and m_A is the Lebesgue measure on A):
  (a) $\mu = \frac{1}{3}\delta_2 + \frac{2}{3}\delta_3$
  (b) $\mu = \frac{1}{3}\delta_2 + \frac{1}{3}\delta_3 + \frac{1}{3}m_{[0,1]}$

(2) Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function and α is the corresponding distribution. Prove that:
  (a) * The set of discontinuity points D of F is countable.
  (b) If F is continuous, then $\alpha(\{x\}) = 0$.

(3) Prove that convergence in probability (i.e. convergence in measure for a probability measure) implies convergence in distribution.

(4) Show that if X is a constant random variable, then $X_n \to X$ in probability is equivalent to $X_n \to X$ in distribution.

(5) Find the characteristic functions of the following discrete distributions $\alpha(A) = P(X \in A)$, $A \in \mathcal{B}(\mathbb{R})$:
  (a) Degenerate (or Dirac or atomic) distribution

$$\alpha(A) = \begin{cases} 1, & a \in A \\ 0, & \text{o.c.} \end{cases}$$

  where $a \in \mathbb{R}$.
  (b) Binomial distribution with $n \in \mathbb{N}$:

$$\alpha(\{k\}) = C_k^n \, p^k (1-p)^{n-k}, \quad 0 \leq k \leq n.$$

  (c) Poisson distribution with $\lambda > 0$:

$$\alpha(\{k\}) = \frac{\lambda^k}{k! e^\lambda}, \quad k \in \mathbb{N} \cup \{0\}.$$

  This describes the distribution of 'rare' events with rate λ.
  (d) Geometric distribution with $0 < p < 1$:

$$\alpha(\{k\}) = (1-p)^k p, \quad k \in \mathbb{N} \cup \{0\}.$$

This describes the distribution of the number of unsuccessful attempts preceding a success with probability p.

(e) Negative binomial distribution

$$\alpha(\{k\}) = C_k^{n+k-1}(1-p)^k p^n, \quad k \in \mathbb{N} \cup \{0\}.$$

This describes the distribution of the number of accumulated failures before n successes. *Hint*: Recall the Taylor series of $\frac{1}{1-x} = \sum_{i=0}^{+\infty} x^i$ for $|x| < 1$. Differentiate this n times and use the result.

(6) Find the characteristic functions of the following absolutely continuous distributions $\alpha(A) = P(X \in A) = \int_A f(x)\,dx$, $A \in \mathcal{B}(\mathbb{R})$ where f is the density function:

(a) Uniform distribution on $[a, b]$

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{o.c.} \end{cases}$$

(b) Exponential distribution

$$f(x) = e^{-x}, \quad x \geq 0$$

(c) The two-sided exponential distribution

$$f(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}$$

(d) The Cauchy distribution

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

Hint: Use the residue theorem of complex analysis.

(e) The normal (Gaussian) distribution with mean μ and variance $\sigma^2 > 0$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$