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Stochastic Interest Rate Models

2.1. CONTINUOUS TIME FINANCE RECAP

- **Stochastic process** – any variable whose value changes over time in an uncertain way => different random trajectories for the variable.

- Discrete vs continuous time stochastic processes:
  - Discrete – the variable value can change only at certain fixed points in time
  - Continuous – changes can take place at any time

- Continuous vs discrete variables:
  - Discrete – only certain values are possible
  - Continuous – can take any value within a certain range

- Continuous-variable, continuous-time – variables can assume any value and changes can occur at any time.

STOCHASTIC PROCESSES

- **Continuous-variable, continuous-time stochastic processes are key to understanding the pricing of options and other derivatives.**

- However, **in practice, most asset prices do not follow continuous-variable, continuous-time stochastic processes.**

- For instance, stock prices are restricted to discrete values (e.g. multiples of a cent) and changes can be observed only when the markets are open.

- Nonetheless, continuous-variable, continuous-time stochastic processes are useful for many valuation purposes.

STOCHASTIC PROCESSES

- **Markov Stochastic Process** – stochastic process where only the current value of a variable is relevant for predicting the future => all past information is irrelevant, as it is already incorporated into today's stock price (weak form of market efficiency, while the strong form states that all relevant information is incorporated in current prices).

- The probability distribution at any particular future time is independent from the path followed by the variable in the past.
- If the weak form of market efficiency were not true, market participants could make above-average returns by interpreting the past behavior of asset prices.

STOCHASTIC PROCESSES

- Assuming a Markov process X(t), the 1-year change ~ $\mathcal{N}(0,1)$.

- 2-year change = $\mathcal{N}(0,1)$ + $\mathcal{N}(0,1)$ = $\mathcal{N}(0,2)$, as both distributions are independent - given that this is a Markov process, the second distribution does not depend on the first.

Δt (very small period of time) change ~ $\mathcal{N}(0, \Delta t)$

WIENER PROCESS

A stochastic process z follows a **Wiener process (or the continuous random walk)** if it has the following properties:

Property 1. *The change Δz during a small period of time Δt is*

$$\Delta z = \epsilon \sqrt{\Delta t} \quad \textbf{(14.1)}$$

where ϵ has a standard normal distribution $\phi(0, 1)$.

Property 2. *The values of Δz for any two different short intervals of time, Δt, are independent.*

It follows from the first property that Δz itself has a normal distribution with

$$\text{mean of } \Delta z = 0$$
$$\text{standard deviation of } \Delta z = \sqrt{\Delta t}$$
$$\text{variance of } \Delta z = \Delta t$$

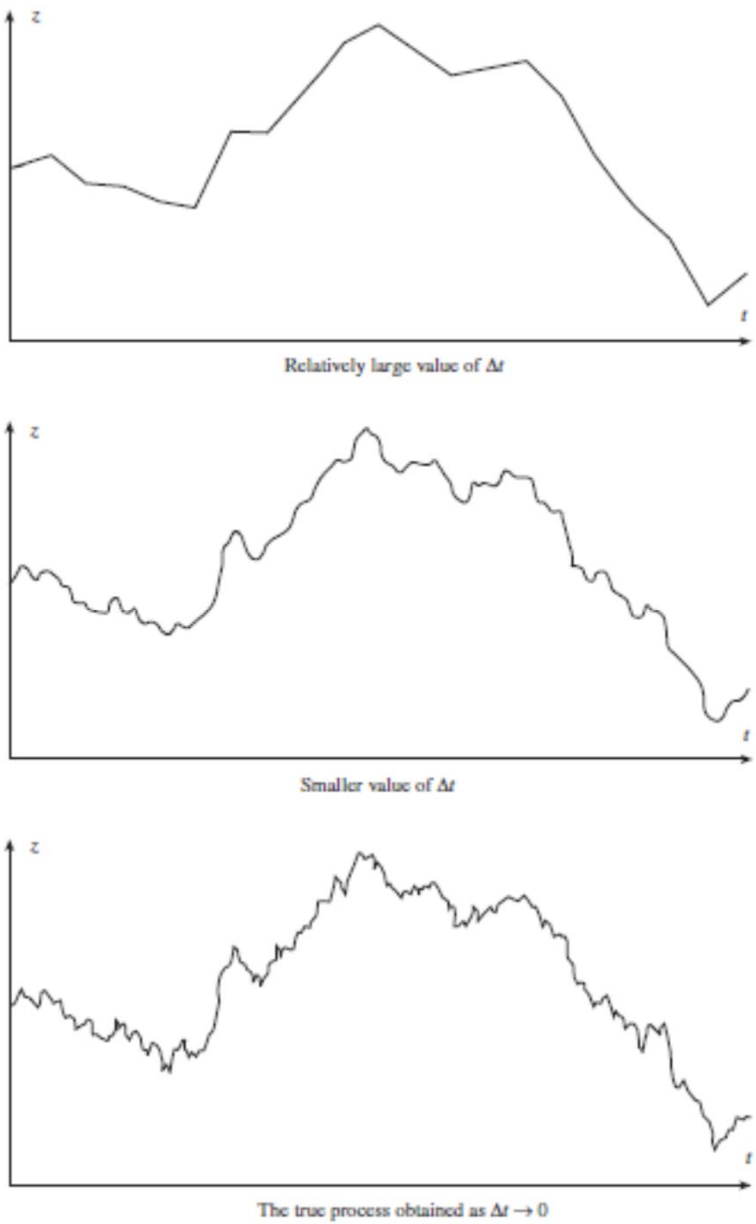
The second property implies that z follows a Markov process.

Source: Hull, John (2015), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 9th Edition

- Therefore , a Wiener process is a Markov process with its change having:
  - **mean (drift) = 0** => the expected value of any future outcome is equal to the current value (Martingale): z=25 => 1 year after, z ~ N(25,1); 5 years after, z ~ N(25,5)
  - **variance (variance rate) = 1** => uncertainty (standard-deviation) is proportional to the square root of time.

WIENER PROCESS

Wiener processes for different magnitudes of change in time:



When Δt -> 0, the path becomes much more irregular, as the size of the movement in the variable in time Δt is proportional to the $\sqrt{\Delta t}$. When Δt is small, $\sqrt{\Delta t}$ is much larger than Δt => the changes in z will be much larger than Δt.

Source: Hull, John (2015), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 9th Edition

Generalized Wiener Process

- Instead of a drift = 0 and a variance rate =1 as in the Wiener process, we may have a generalized wiener process, where the drift can assume any value ***a*** and the variance rate can be b^2 => **Generalized Wiener Process**.

$$dx = a\,dt + b\,dz$$ where a and b are constants.

- For very small time changes Δt: $\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$

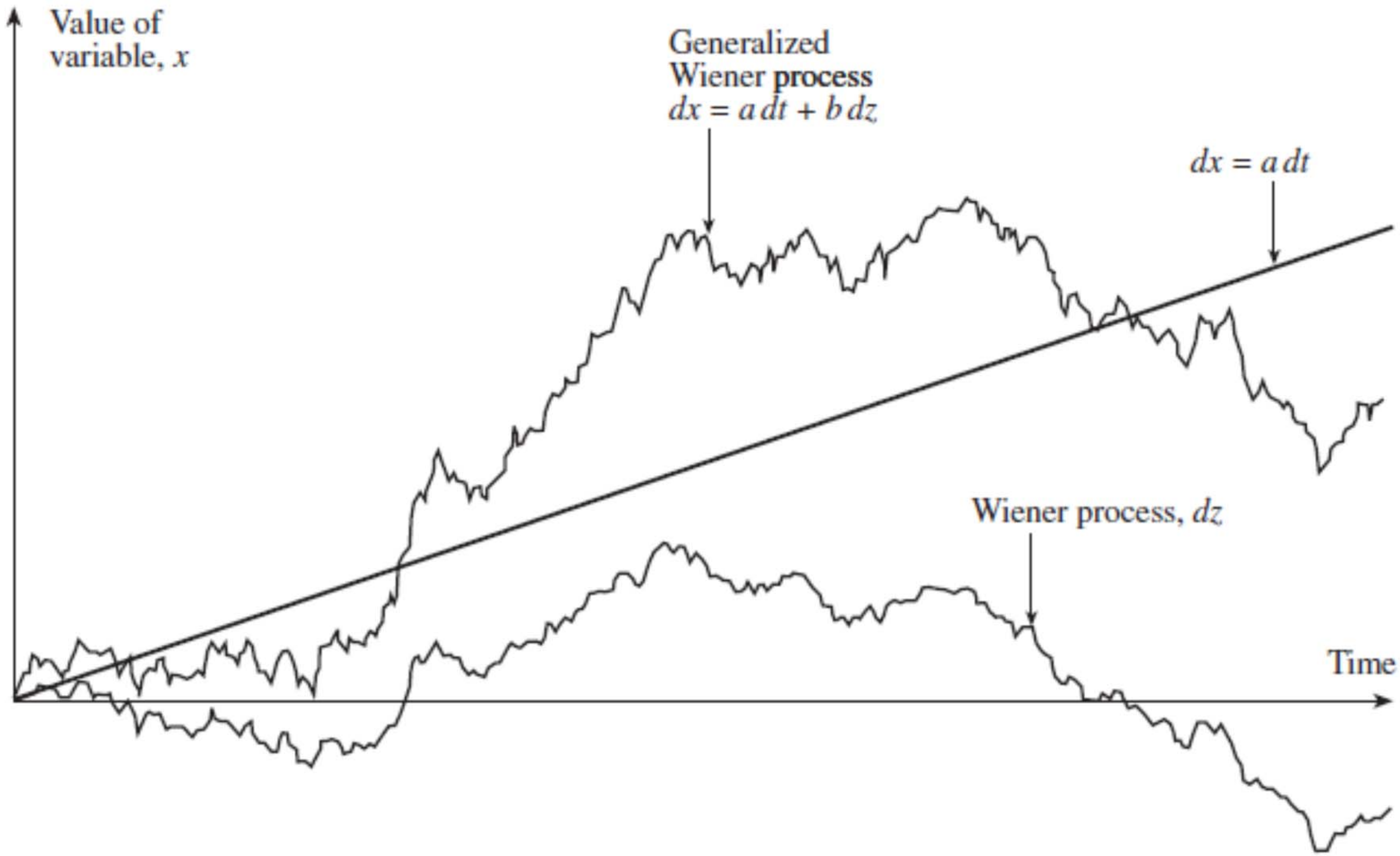
Δx ~ N, with

$$\text{mean of } \Delta x = a\,\Delta t$$
$$\text{standard deviation of } \Delta x = b\sqrt{\Delta t}$$
$$\text{variance of } \Delta x = b^2\Delta t$$

- The average increases are proportional to time (if there is no drift, the mean doesn't change and is equal to 0).

GENERALIZED WIENER PROCESS

Figure 14.2 Generalized Wiener process with $a = 0.3$ and $b = 1.5$.



Source: Hull, John (2015), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 9th Edition

ITÔ PROCESS

- Definition: Generalized Wiener process with average and standard-deviation as functions of the underlying variable and time (instead of constant along time):

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

- For small time intervals, we may assume that the average and the standard-deviation don't change (we're assuming that the drift and the variance rate don't change between *t* an $t+\Delta t$):

$$\Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$$

- This is still a Markov process, as *a* and *b* only depend on the current value of x, not on previous values.

ITÔ PROCESS

- It may be tempting to assume that a stock price follows a generalized Wiener process (constant drift and variance).
- However, this assumption is not valid, having in mind that investors require or expect a given level of returns (as a % variation) regardless the price level, i.e. for higher prices, expected changes will also be higher.

- One can replace the assumption of constant expected drift by the assumption of constant expected returns (i.e. constant expected drift divided by the stock price ⇔ variable drift along time).

ITÔ PROCESS

- If S is the stock price at time t => expected drift rate in S *(i.e. a(x,t))* must be μS (being μ constant, corresponding to the expected rate of return on the stock, expressed in decimal form).

- In a short interval of time Δt, the expected increase in S is $\mu S \Delta t$, i.e the expected rate of return on the stock, times the stock price, times the time interval:

$$\Delta S = \mu S \Delta t$$

$$\Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$$

- If **Δt -> 0 =>**

$$dS = \mu S dt \Leftrightarrow \frac{dS}{S} = \mu dt$$

- This corresponds to the price of an asset following a continuously compounding process (under no uncertainty, being μ = risk-free rate in a risk-neutral world): $S_T = S_0 e^{\mu T}$

GEOMETRIC BROWNIAN MOTION

- Given that in practice there is uncertainty, a reasonable assumption is that the variability of the percentage return (σ) in a short period of time Δt is the same regardless the stock price.

⬇

- An investor is as uncertain about his return when the stock price is high or low.
- Accordingly, the standard deviation of the change in a short period of time must be proportional to the stock price, as the standard deviation for the percentage change is constant – **Geometric Brownian Motion**:

$$dS = \mu S dt + \sigma S dz \Leftrightarrow$$

$$\frac{dS}{S} = \mu dt + \sigma dz$$

GEOMETRIC BROWNIAN MOTION

- Example:

Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding. In this case, $\mu = 0.15$ and $\sigma = 0.30$. The process for the stock price is

$$\frac{dS}{S} = 0.15\,dt + 0.30\,dz$$

If S is the stock price at a particular time and ΔS is the increase in the stock price in the next small interval of time, the discrete approximation to the process is

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\epsilon\sqrt{\Delta t}$$

where ϵ has a standard normal distribution. Consider a time interval of 1 week, or 0.0192 year, so that $\Delta t = 0.0192$. Then the approximation gives

$$\frac{\Delta S}{S} = 0.15 \times 0.0192 + 0.30 \times \sqrt{0.0192}\,\epsilon$$

or

$$\Delta S = 0.00288S + 0.0416S\epsilon$$

Source: Hull, John (2015), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 9th Edition

Itô's Lemma

- An option price (*G*) is a function of the underlying asset's price and time.
- Therefore, it is important to understand the behavior of functions of stochastic variables.
- An important result was discovered by K. Itô in 1951 and is known as **Itô's lemma**.
- Assuming that a variable x follows an Itô process:

$$dx = a(x, t)\,dt + b(x, t)\,dz$$

where dz is a Wiener process and a and b are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2. Itô's lemma shows that a function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$

Itô's Lemma

- Thus, G also follows an Itô process with a drift rate

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

and a variance rate of

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2$$

$$dx = a(x, t)\,dt + b(x, t)\,dz$$

- Assuming that the stock price follows a Geometric Brownian Motion, with constant μ and σ:

$$dS = \mu S dt + \sigma S dz$$

- From Ito's Lemma it follows that

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S\,dz$$

- … in line with $dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$

ITÔ'S LEMMA

- Therefore, both *S* and *G* are affected by the same volatility source – *dz*.

- This is in line with the Black-Scholes option pricing formula, as G (the option price) is determined by the instantaneous volatility of the returns of the underlying asset price.

PROBABILITY DISTRIBUTION

- From the stochastic process of the rate of returns,

$$\frac{dS}{S} = \mu dt + \sigma dz$$

- Its distribution gets

$$\frac{\Delta S}{S} \sim \phi(\mu \Delta t, \sigma^2 \Delta t)$$

- Assuming $G = \ln S$, since $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, $\frac{\partial G}{\partial t} = 0$, it follows from the Itô's lemma that

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dz$$

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S\, dz$$

Probability Distribution

Since μ and σ are constant, this equation indicates that $G = \ln S$ follows a generalized Wiener process. It has constant drift rate $\mu - \sigma^2/2$ and constant variance rate σ^2. The change in $\ln S$ between time 0 and some future time T is therefore normally distributed, with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$. This means that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

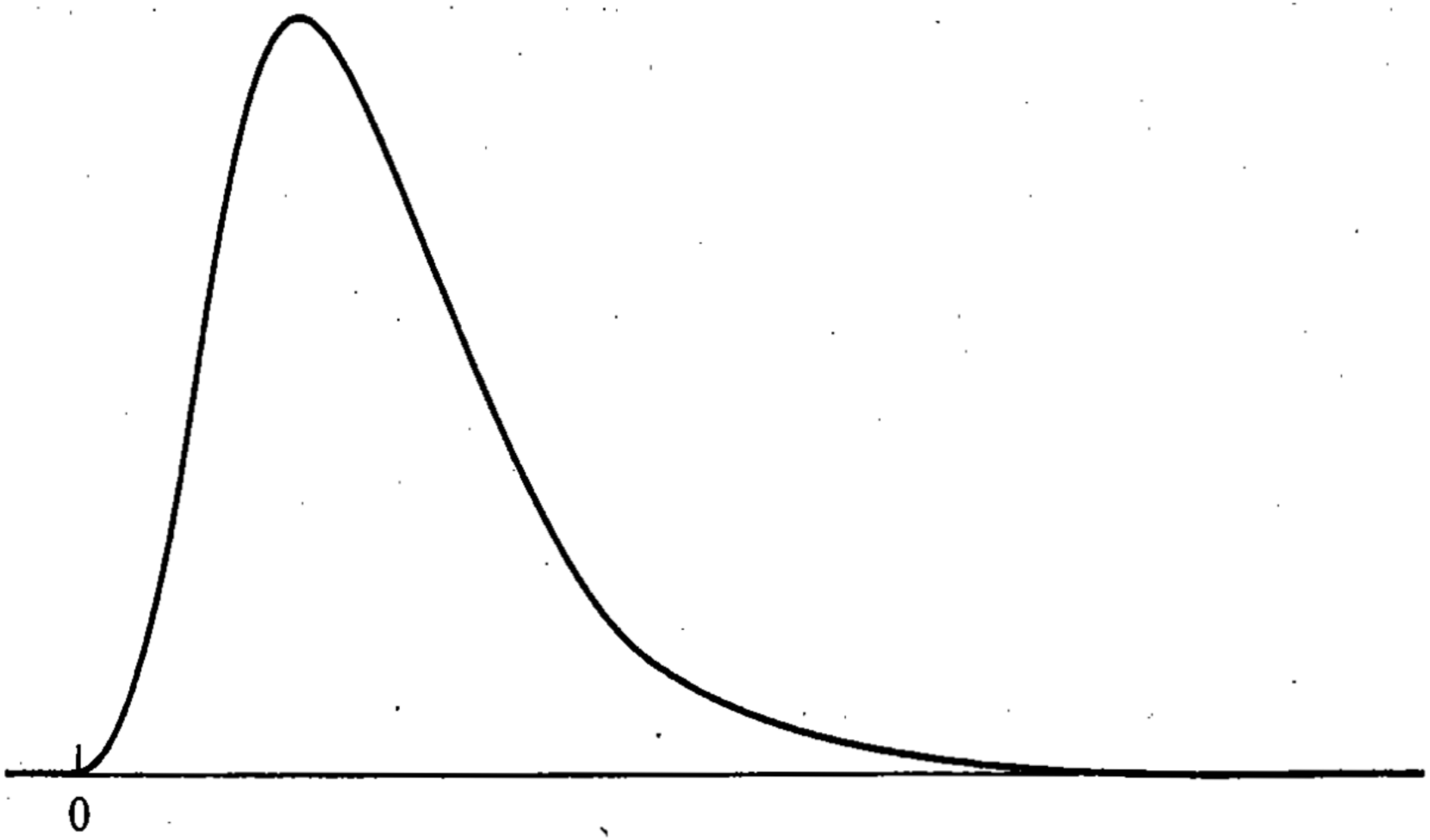
or

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

- This equation shows that ln S_T is normally distributed (and S_T has a log normal distribution), with a standard deviation $\sigma\sqrt{T}$ that is proportional to the square root of time.

PROBABILITY DISTRIBUTION

Figure 13.1 Lognormal distribution.



Source: Hull, John (2009), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 7th Edition