

3.

Return Generating Models

3.1 Estimation versus model risk

- Learning Objectives
- Estimation Risk
- Model Risk
- Questions

3. Return Generating Models

- Estimation risk versus model risk
- Constant correlation models
- Single-factor models
- Fundamental analysis
- Multi-factor models

Learning objectives

- state how much data is needed to perform mean-variance portfolio analysis,
- discuss the problem with obtaining the data,
- be able to explain the notion of *estimation risk* in the context of MVT
- identify the main types of return generating models
- be able to explain the notion of *model risk* associated with return generating models.

Problems with mean-variance analysis

- Suppose we work with n assets.
- To apply the techniques of MVT, we need the $n \times n$ elements of the variance-covariance matrix of the returns, and the n expected returns.
- This requires estimation of

$$\underbrace{n}_{\bar{R}_i} + \underbrace{n}_{\sigma_i} + \underbrace{n(n-1)/2}_{\sigma_{ij}}$$

parameters.

- The number of parameters grows with the square of the number of assets.
 - If $N = 10$, need 65 numbers.
 - If $N = 100$, need 5150 numbers.
 - If $N = 1000$, need 506000 numbers.

Professional estimates

Analysts Approach:

- One could instead use professional estimates.
- Not clear that the technique is reliable.
- You would need an awfully large number of analysts to estimate so many parameters.
- The cost of employing so many analysts would outweigh the benefits.

Where do we get the numbers from?

Historical Approach:

- Use historical time series to estimate the necessary parameters.
- For this to be reliable, you need many more data points than numbers to estimate. Where would we get that much data?
- We are generally interested in 1-year time horizons. How many years back can we go?
- Markets are not qualitatively the same very far back and most companies are not that old or very similar to what they were.
- We could use shorter time horizons and scale. But evidence exists that short term behaviour qualitatively different.

*OBS: MVT inputs are from the future distribution of returns at T .
NOT from the distribution of past realised returns.*

Estimation Risk

Estimation of MVT inputs is only a problem to the extent that MVT results are sensitive to the parameters.

- Unfortunately MVT is extremely sensitive to parameter estimation.
- This is known as **estimation risk**.
- Relatively small estimation errors lead to:
 - Wrong assessment of the investment opportunity set
 - Wrong deduction of tangent portfolios
 - Wrong efficient frontier



Construction of **inefficient** portfolios.

Q: In the presence of estimation risk are we better off applying MVT or just using naive portfolio construction?

Model Risk

Model Approach:

- One could instead rely on the use models .
- Models use simplifying assumptions to reduce the number of the necessary estimates \Rightarrow reduces **estimation risk**.
- However, the assumptions may be nonrealistic – **model risk** – and may also lead to :
 - Wrong assessment of the investment opportunity set
 - Wrong deduction of tangent portfolios
 - Wrong efficient frontier



It may also lead to construction of **inefficient** portfolios

OBS: There is a estimation risk versus model risk tradeoff.

Return Generating Models

We are going to look at three types of Return-generating models:

- **Constant correlation models (CCM)**
- **Single factor models (SFM)**
- **Multi-factor models (MFM)**

Their advantages are:

- Reduction of the number of parameters.
- Allow for a ranking of assets that permits three-step procedures to get tangent portfolios, even when shortselling is not allowed!

Questions

- 1 What are the data problems with mean-variance analysis?
- 2 Explain what is estimation risk.
- 3 Explain what is model risk.
- 4 Why to we have a model versus estimation risk tradeoff when using return generating models for MVT?

3.2 Constant Correlation Models

- Learning objectives
- The average correlation
- Constant correlation models
- Finding tangent portfolios under CCM

Average correlation models

- Biggest amount of parameters to estimate are covariances, or equivalently **correlation coefficients**,
- It would be nice if we did not need to determine the $n(n-1)/2$ different correlation coefficients.
- What if, instead, we focus our estimation process only on the **average correlation**

$$\rho = \frac{\sum_{i=1}^n \sum_{j>i}^n \rho_{ij}}{\frac{n(n-1)}{2}}$$

- And, when applying MVT we assume all correlations equal to ρ .

Constant correlation models

CCM Assumption

We suppose all asset correlations in the market are equal to the average market correlation:

$$\rho_{ij} = \rho \quad \text{for all } i, j \text{ with } i \neq j$$

The number of parameters to estimate reduces to

- $\bar{R}_i, i = 1, 2, \dots, n$
- $\sigma_i, i = 1, 2, \dots, n$
- ρ
- That accounts to $2n + 1$ parameters.
- So, for constant correlation models, the number of parameters grows linearly with the number of assets, rather than quadratically.

CCM: ranking of risky assets

- In general, MVT does not allow for any ranking of risky assets.
- The good news is that – if we were willing to accept the constant correlation assumption – ranking of risky assets is possible.

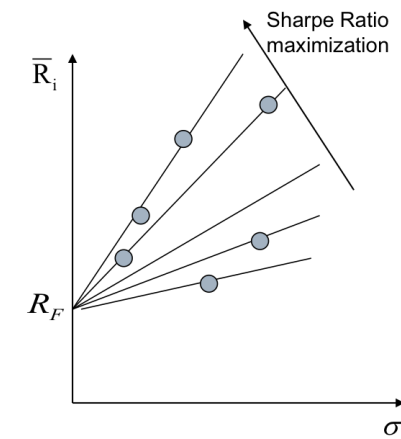
Theorem

Under the CCM assumption, if a given risky asset belongs to the tangent portfolio, then all risky assets with higher Sharpe Ratio also belong to the tangent portfolio.

- This result considerably simplifies the calculations for finding **tangent portfolios**, even when shortselling is not allowed.
⇒ It all reduces to a **three-step procedure**.

Finding Tangent Portfolios

Step 1: ranking by $SR_i = \frac{\bar{R}_i - R_f}{\sigma_i}$



Example: step 1

➤ Example:

$$R_F = 5\%$$

$$\rho = 0.5$$

Average correlation

| Security No. i | Expected Return \bar{R}_i | Excess Return $\bar{R}_i - R_F$ | Standard Deviation σ_i | Excess Return to Standard Deviation $\frac{\bar{R}_i - R_F}{\sigma_i}$ |
|---------------------|--------------------------------|------------------------------------|----------------------------------|---|
| 1 | 29 | 24 | 3 | 8.0 |
| 2 | 19 | 14 | 2 | 7.0 |
| 3 | 29 | 24 | 4 | 6.0 |
| 4 | 35 | 30 | 6 | 5.0 |
| 5 | 14 | 9 | 2 | 4.5 |
| 6 | 21 | 16 | 4 | 4.0 |
| 7 | 26 | 21 | 6 | 3.5 |
| 8 | 14 | 9 | 3 | 3.0 |
| 9 | 15 | 10 | 5 | 2.0 |
| 10 | 9 | 4 | 2 | 2.0 |
| 11 | 11 | 6 | 4 | 1.5 |
| 12 | 8 | 3 | 3 | 1.0 |

Finding Tangent Portfolios

Step 2: cut-off

- If shortselling is allowed, we know all assets belong to the tangent portfolio and

$$C^* = C_n = \frac{\rho \sum_{i=1}^n \left(\frac{\bar{R}_i - R_F}{\sigma_i} \right)}{1 - \rho + n\rho}$$

- C^* is called the **cut-off level** and it tells us to take:
 - long positions in all assets with $SR_i > C^*$,
 - no investment if it happens $SR_i = C^*$, and
 - short positions in all assets with $SR_i < C^*$.

Finding Tangent Portfolios

Step 2: cut-off

- If shortselling is **NOT** allowed, we do not know how many assets belong to T . So, in **Step 2**, we need to proceed iteratively, starting from the asset with the highest SR and moving downwards.
- We start by considering T has only one asset, $k = 1$, then $k = 2$, $k = 3$, etc.
- The cut-off C^* is defined

$$C^* = C_k = \frac{\rho \sum_{i=1}^k \left(\frac{\bar{R}_i - R_F}{\sigma_i} \right)}{1 - \rho + k\rho} \text{ for } k \text{ s.t. } \begin{cases} SR_i > C^* & i = 1, \dots, k \\ SR_i < C^* & i = k + 1, \dots, n \end{cases}$$

and we stop at the first k that verifies the **condition** above.

- Only assets with $SR_i > C^*$ are included in the tangent portfolio T , so, k is its number of assets.

Example: step 2

➤ Example (cont)

| Security No. i | $\frac{\rho}{1 - \rho + i\rho}$ | $\sum_{j=1}^i \frac{\bar{R}_j - R_F}{\sigma_j}$ | C_i | $\frac{R_i - R_F}{\sigma_i}$ |
|---------------------|---------------------------------|---|--------------------------|------------------------------|
| 1 | $\frac{1}{2}$ | 8 | $\frac{8}{2} = 4$ | 8 |
| 2 | $\frac{1}{3}$ | 15 | $\frac{15}{3} = 5$ | 7 |
| 3 | $\frac{1}{4}$ | 21 | $\frac{21}{4} = 5.25$ | 6 |
| 4 | $\frac{1}{5}$ | 26 | $\frac{26}{5} = 5.2$ | 5 |
| 5 | $\frac{1}{6}$ | 30.5 | $\frac{30.5}{6} = 5.08$ | 4.5 |
| 6 | $\frac{1}{7}$ | 34.5 | $\frac{34.5}{7} = 4.93$ | 4 |
| 7 | $\frac{1}{8}$ | 38 | $\frac{38}{8} = 4.75$ | 3.5 |
| 8 | $\frac{1}{9}$ | 41 | $\frac{41}{9} = 4.56$ | 3 |
| 9 | $\frac{1}{10}$ | 43 | $\frac{43}{10} = 4.3$ | 2 |
| 10 | $\frac{1}{11}$ | 45 | $\frac{45}{11} = 4.09$ | 2 |
| 11 | $\frac{1}{12}$ | 46.5 | $\frac{46.5}{12} = 3.88$ | 1.5 |
| 12 | $\frac{1}{13}$ | 47.5 | $\frac{47.5}{13} = 3.65$ | 1 |

$C^* = 5.25$
(shortselling not allowed)

$C^* = 3.65$
(shortselling allowed)

Finding Tangent Portfolios

Step 3: composition

- Given the cut-off level, we have the following formula for the entries of our usual vector Z

$$z_i = \frac{1}{(1-\rho)\sigma_i} \left(\frac{\bar{R}_i - R_f}{\sigma_i} - C^* \right),$$

note C^* differs depending on whether we can or cannot shortsell.

- Recall $Z = \lambda X_T$, for λ constant and X_T the vector of weights T ,

- Shortselling allowed:
$$x_i^T = \frac{z_i}{\sum_{j=1}^n z_j}$$
- Shortselling allowed *a la* Lintner:
$$x_i^T = \frac{z_i}{\sum_{j=1}^n |z_j|}$$
- Shortselling **not** allowed:
$$x_i^T = \frac{z_i}{\sum_{j=1}^k z_j}$$

Example: step 3

For our example we get

$$X_T = \begin{pmatrix} 89.54\% \\ 103.41\% \\ 36.25\% \\ 13.87\% \\ 26.15\% \\ 5.35\% \\ -1.58\% \\ -13.47\% \\ -20.44\% \\ -51.11\% \\ -33.28\% \\ -54.68\% \end{pmatrix} \quad X_T^{\text{Lintner}} = \begin{pmatrix} 19.94\% \\ 23.02\% \\ 8.07\% \\ 3.09\% \\ 5.82\% \\ 1.19\% \\ -0.35\% \\ -3.00\% \\ -4.55\% \\ -11.38\% \\ -7.41\% \\ -12.17\% \end{pmatrix} \quad X_T^{\text{no short}} = \begin{pmatrix} 46.32\% \\ 44.21\% \\ 9.47\% \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example: model risk

Consider the following information concerning three risk assets:

| | \bar{R} | σ | Correlations | | |
|---|-----------|----------|--------------|------|-----|
| A | 10% | 7% | 1 | -0.4 | 0.7 |
| B | 15% | 15% | -0.4 | 1 | 0 |
| C | 20% | 30% | 0.7 | 0 | 1 |

For simplicity assume short selling is allowed. Consider the lending rate $R_f^l = 3\%$ and the borrowing rate $R_f^a = 7\%$. Determine:

- The efficient frontier and the two tangent portfolios T and T_2 .
- Repeat the same task under the Constant Correlation Model (CCM) assumption.
- Compare and assess model risk in the context of this example.

Example model risk: tangent portfolios

In this Example the average correlation is $\bar{\rho} = 0.1$, so for the CCM that is the constant correlation parameter.

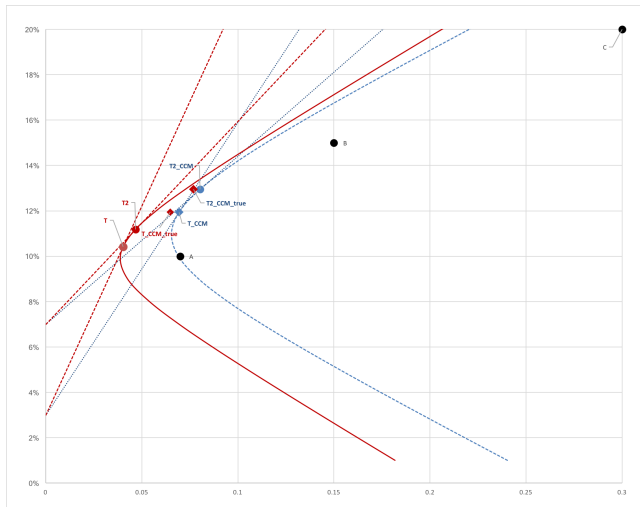
- Tangent Portfolios

$$X_T = \begin{pmatrix} 82.31\% \\ 27.01\% \\ -9.32\% \end{pmatrix} \quad X_T^{\text{CCM}} = \begin{pmatrix} 68.60\% \\ 24.37\% \\ 7.37\% \end{pmatrix}$$

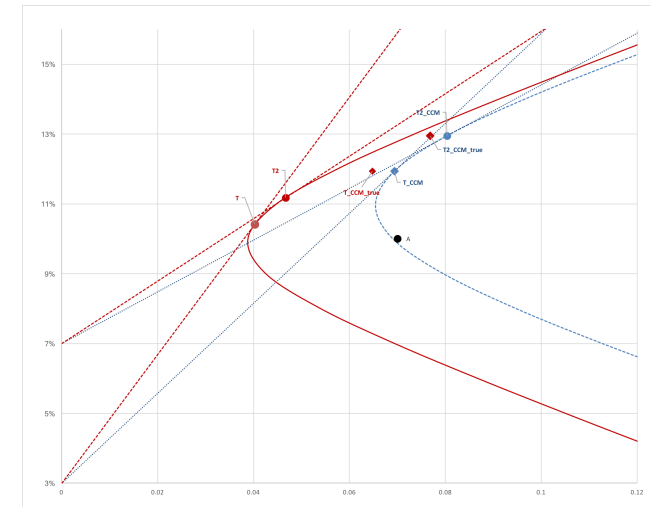
$$X_{T_2} = \begin{pmatrix} 72.19\% \\ 32.05\% \\ -4.24\% \end{pmatrix} \quad X_{T_2}^{\text{CCM}} = \begin{pmatrix} 53.76\% \\ 33.50\% \\ 12.74\% \end{pmatrix}$$

It is clear the tangent portfolios are not the same, under the true correlation structure and assuming a constant correlation. Indeed, using the CCM we would end up recommending non-efficient portfolios.

Example model risk: EF



Example: model risk: EF zoom



Questions

- What is the main assumption underlying CCMs?
- What is the total number of parameters one needs to estimate when using a CCM?
- When is it appropriate to use a CCM?
- When we use the average correlation as the constant correlation in a CCM, is it possible to rank individual assets? How?
- Explain the three-step procedure to find tangent portfolios.

3.3 Single Factor Models

- Learning objectives
- One-factor models
- The beta
- Parameter Estimation
- Finding Tangent Portfolios
- Questions

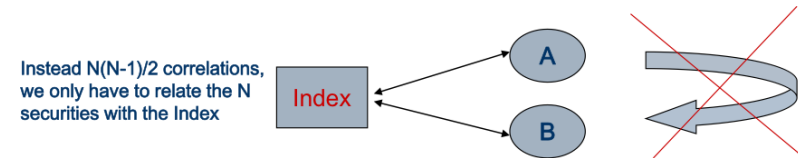
Learning objectives

- define a single factor model mathematically,
- compare the amount of data required for a single factor model with the general MVT setup,
- derive expected return, variance, and covariance.
- define specific, systematic and diversifiable risk,
- discuss and compute variances of large portfolios. factor models.
- Find alpha and beta given times series of returns.
- Derive the relationship between idiosyncratic risk and beta estimation.
- Discuss and use Blume's technique for improving beta estimation.

One-factor models

A very popular approach is to use a **one-factor model** in which all correlated movement comes a single source.

- One-factor means one **common** factor.
- In practice, many times this factor is an index.



- This is quite different from having **one factor in total**.

Mathematical formulation of one-factor model

- We set

$$R_i = \alpha_i + \beta_i R_m + e_i,$$

where R_m is the return on the market. α_i, β_i , are constants and the variables e_i have mean zero, and are uncorrelated with the market.

- This does not say anything until we make the crucial additional assumption:

$$\mathbb{E}(e_i e_j) = 0, \text{ for } i \neq j,$$

that is the variables e_i are not correlated with each other.

SFM – Assumption

There is a common factor that is able of capturing all possible dependence between any two assets, and we are able to identify it.

Data reduction

We now have to estimate the parameters:

- $\alpha_i, i = 1, 2, \dots, n$
- $\beta_i, i = 1, 2, \dots, n$
- $\mathbb{E}(R_m)$,
- $\text{Var}(R_m)$,
- $\sigma_{e_i}, i = 1, 2, \dots, n$.
- That accounts to $3n + 2$ parameters.
- For one-factor models, the number of parameters grows linearly with the number of assets, rather than quadratically.
- The zero correlation assumption for the terms e_i vastly reduces the amount of data required.

Essential results for single-factor models

$$1 \quad \bar{R}_i = \alpha_i + \beta_i \bar{R}_m$$

$$\begin{aligned} \mathbb{E}(R_i) &= \mathbb{E}(\alpha_i + \beta_i R_m + e_i), \\ &= \alpha_i + \beta_i \mathbb{E}(R_m) + \underbrace{\mathbb{E}(e_i)}_0, \\ &= \alpha_i + \beta_i \bar{R}_m, \end{aligned}$$

$$2 \quad \sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2.$$

$$\begin{aligned} \text{Var}(R_i) &= \text{Var}(\alpha_i + \beta_i R_m + e_i) \\ &= \text{Var}(\beta_i R_m) + \text{Var}(e_i), \\ &= \beta_i^2 \text{Var}(R_m) + \text{Var}(e_i) \\ &= \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2 \end{aligned}$$

Risk division

We have divided the risk of asset i into two pieces.

$$\sigma_i^2 = \underbrace{\beta_i^2 \sigma_m^2}_{\text{systematic risk}} + \underbrace{\sigma_{e_i}^2}_{\text{specific risk}}$$

- The first part arises from exposure to the market and is called **systematic risk**.
- The remaining part is called **specific risk**, and is the part unique to the security and can be diversified away. It is also called :
 - alpha risk,
 - diversifiable risk,
 - unsystematic risk,
 - residual risk.

Covariance

$$3 \quad \sigma_{ij} = \beta_i \beta_j \sigma_m^2, \text{ for } i \neq j.$$

$$\begin{aligned} \text{Cov}(R_i, R_j) &= \mathbb{E}((R_i - \bar{R}_i)(R_j - \bar{R}_j)), \\ &= \mathbb{E}((\beta_i(R_m - \bar{R}_m) + e_i)(\beta_j(R_m - \bar{R}_m) + e_j)) \\ &= \beta_i \beta_j \mathbb{E}((R_m - \bar{R}_m)^2) + \beta_i \underbrace{\mathbb{E}((R_m - \bar{R}_m)e_j)}_0 \\ &\quad + \beta_j \underbrace{\mathbb{E}((R_m - \bar{R}_m)e_i)}_0 + \underbrace{\mathbb{E}(e_i e_j)}_0. \end{aligned}$$

For $i \neq j$, the last 3 terms are zero from our assumption that e_i are uncorrelated with the market and each other.

Beta

Clearly, for $i = m$

$$R_m = 0 + 1.R_m,$$

So the market portfolio has

$$\begin{aligned} \alpha_m &= 0, \\ \beta_m &= 1. \end{aligned}$$

OBS: So the level of beta is a measure of risk level compared to the common factor - systematic risk measure.

Using beta - example

- Suppose a stock i has a beta of $\beta_i = 3$.
- Suppose the market goes up 1%.
- What can we say about the stock's return?

$$R_i = \alpha_i + \beta_i R_m + e_i$$

- Assuming everything else constant, our best estimate without further knowledge is:

$$\frac{\partial R_i}{\partial R_m} = \beta_i$$

$$dR_i = \beta_i dR_m$$

i.e., R_i is expected to go up by 3% on average.

OBS: However, any value is possible because of specific part.

Risk of portfolios

Let us now consider a portfolio of n assets, with weights x_i . Its variance is

$$\begin{aligned} \sigma_P^2 &= \text{Var} \left(\sum_{i=1}^n x_i R_i \right) = \mathbb{E} \left(\left(\sum_{i=1}^n x_i (R_i - \bar{R}_i) \right)^2 \right) \\ &= \mathbb{E} \left(\left(\sum_{i=1}^n x_i \beta_i (R_m - \bar{R}_m) + \sum_{i=1}^n x_i e_i \right)^2 \right), \\ &= \mathbb{E} \left(\left(\sum_{i=1}^n \beta_i x_i \right)^2 (R_m - \bar{R}_m)^2 \right) + \sum_{i=1}^n x_i^2 \mathbb{E}(e_i^2), \\ &= \left(\sum_{i=1}^n x_i \beta_i \right)^2 \sigma_m^2 + \sum_{i=1}^n x_i^2 \sigma_{e_i}^2 \\ &= (\beta_P)^2 \sigma_m^2 + \sigma_{eP}^2 \end{aligned}$$

Beta of large portfolios

- Let us consider large homogeneous portfolios.
- In that case, if we consider n assets are $x_i = \frac{1}{n}$ for $i = 1, \dots, n$
- An the homogenous portfolio variance is

$$\begin{aligned} \sigma_H^2 &= \left(\sum_{i=1}^n \frac{1}{n} \beta_i \right)^2 \sigma_m^2 + \sum_{i=1}^n \left(\frac{1}{n} \right)^2 \sigma_{e_i}^2 \\ &= \frac{\sum_{i=1}^n \beta_i}{n} + \frac{1}{n} \frac{\sum_{i=1}^n \sigma_{e_i}^2}{n} \\ &= (\bar{\beta})^2 \sigma_m^2 + \frac{1}{n} \bar{\sigma}_{e_i}^2, \end{aligned}$$

where the term $\bar{\beta}$ is the average beta and $\bar{\sigma}_{e_i}^2$ is the average specific variance.

- As $n \rightarrow \infty$ we get, the second term goes to zero individual specific risks are bounded,

$$\lim_{n \rightarrow \infty} \sigma_H^2 = (\bar{\beta})^2 \sigma_m^2$$

Large portfolio limit

- By using a large portfolio of many different stocks, one can make the **diversifiable risk** disappear (which is why it's called that.)

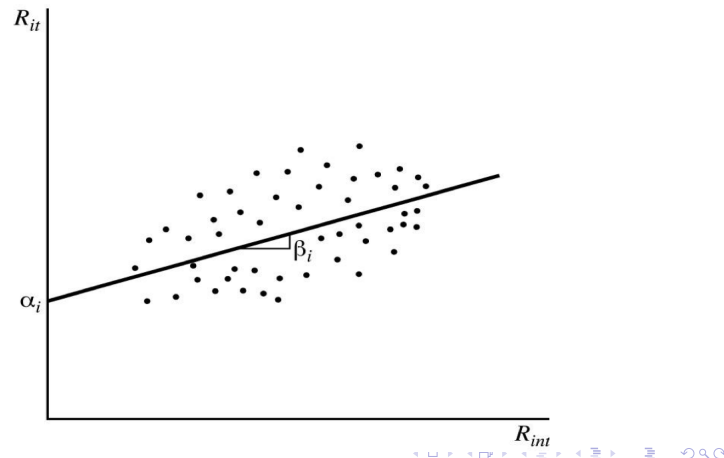
| Number of Securities | Residual Risk (Variance) Expressed as a Percent of the Residual Risk Present in a One-Stock Portfolio with $\sigma_{e_i}^2$ a Constant |
|----------------------|--|
| 1 | 100 |
| 2 | 50 |
| 3 | 33 |
| 4 | 25 |
| 5 | 20 |
| 10 | 10 |
| 20 | 5 |
| 100 | 1 |
| 1000 | 0.1 |

- The beta part, the **undiversifiable risk**, does not disappear.
- We expect risk premia for taking **beta risk**, but not diversifiable risk.

Parameter Estimation

We need to estimate the parameters α_i, β_i and $\sigma_{\epsilon_i}^2$, knowing

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i$$



Using linear regression

- Once we have β_i , we use $\mathbb{E}(R_i - (\alpha_i + \beta_i R_m)) = 0$, to determine α_i

$$\alpha_i = \mathbb{E}(R_i - \beta R_m) = \frac{1}{T} \sum_{t=1}^T (R_{i,t} - \beta R_{m,t}).$$

- Finally to get $\sigma_{\epsilon_i}^2$, we use both α_i, β_i we can determine all ϵ_{it}

$$\epsilon_{it} = R_{i,t} - (\alpha_i + \beta_i R_{m,t})$$

and

$$\sigma_{\epsilon_i}^2 = \text{Var}(\epsilon_i) = \frac{\sum_{t=1}^T (\epsilon_{i,t})^2}{T}$$

Using linear regression

The historical Approach:

- Let $R_{i,t}$ and $R_{m,t}$ denote the returns of asset i and the market m for the t^{th} period.
- Take the sample size be T , i.e., $t = 1, \dots, T$
- Compute average returns of asset i , \bar{R}_i , and market, \bar{R}_m .
- Find $\text{Cov}(R_i, R_m) = \frac{1}{T} \sum_{t=1}^T (R_{i,t} - \bar{R}_i)(R_{m,t} - \bar{R}_m)$.
- And $\sigma_m^2 = \text{Var}(R_m) = \frac{1}{T} \sum_{t=1}^T (R_{m,t} - \bar{R}_m)^2$
- Then, β_i is determined by

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}.$$

Commentary

- Linear regression almost always works, in the sense that it will always find a straight line through a cloud of points that minimizes the least-squares error.
- The residual generated will be uncorrelated with the market and will have zero expectation.
- However, the residuals may well have high correlation with each other.
- The model says that the residuals are uncorrelated with each other.
- So when they are highly correlated the model is a poor fit to the data, and one should consider carefully whether it should be used.

How good are the estimates?

- It's important to realize that the estimates can contain a lot of noise.
- Even if the model was wholly correct, our computation of the values of α and β would contain errors – **estimation risk**.
- Suppose we start with a **true model** with parameters as follows

| | |
|--------------------------|-------|
| α | 1%, |
| β | 2, |
| R_m standard deviation | 2.5%, |
| e_i standard deviation | 2%, |
| R_m mean | 3%, |

and generate a time series – a possible sample – and then measure the implied α and β .

Q: What happens?

Many times

Suppose we do this to get many different samples:

| α^{sample} | β^{sample} |
|-------------------|------------------|
| 1.17 | 1.93 |
| 1.77 | 1.75 |
| 1.51 | 1.78 |
| 1.91 | 1.72 |
| 1.57 | 1.78 |
| 1.76 | 1.81 |
| 2.37 | 1.56 |
| 2.33 | 1.51 |
| 0.06 | 2.34 |
| 1.12 | 2.03 |
| -1.08 | 2.65 |
| -6.50 | 4.51 |

Synthetic data

We give such a series:

| market return | stock return |
|---------------|--------------|
| 6.430% | 14.48% |
| 3.998% | 8.89% |
| 2.350% | 5.57% |
| 0.506% | 2.43% |
| 2.431% | 5.44% |
| 8.445% | 14.79% |
| 0.624% | -1.49% |
| 2.484% | 11.01% |
| 0.151% | 3.08% |

Using this sample we get:

$$\alpha_i^{sample} = 1.73\% \quad , \quad \beta_i^{sample} = 1.77 .$$

Estimation risk

- We see that there is a lot of variation. Some of the values of β are quite good but some are terrible. You would never know which is the case with real data. → there is **estimation risk**, even within a model.
- How can we get less noisy estimates? We need more data points. We can use monthly returns instead of yearly returns to estimate α and β . This will give us less noise but will provide a good estimate of month on month correlation rather than year on year correlation.
- These will be similar but there is evidence that they are different.

Portfolio effects

- The size of the idiosyncratic (specific) risk is an important factor in determining the accuracy of estimations of beta.
- In particular, if we take a portfolio that has had much of the idiosyncratic risk diversified away, the beta estimate will be much better.
- The more similar to the portfolio is to the market, the less residual risk there will be.

Blume Adjustment

If the distribution of observed betas is too wide then

- observed betas that are high should be overestimates,
- observed betas that are low should be underestimates

Measure betas of stocks in one period, β_1 , and then remeasure them for the same stocks in the next period, β_2 .

If we are correct then high betas should tend to go down, and low betas should tend to go up.

If we regress to get

$$\beta_2 = a\beta_1 + b,$$

then we should get $a < 1$.

Beta bias

- If we observe a large sample of stocks and compute the beta for each, we will get a distribution of betas.
- Each of these betas will have an error.
- If the observed beta is very high then the high value is as likely to come from upwards error as from the true value being high.
- Similarly, for very low betas, it is as likely to come from a large downwards error as from the true value being low.
- This suggests that betas should have a tighter distribution than typically observed.

OBS: True Betas should be closer to one than observed ones.

Blume Adjustment

- Blume measured the value of β for a range of stocks in a number of periods.
- He then regressed the beta for each stock in each period against its beta in the preceding period.
- He found in each case that the straight line had slope less than one.

| Period | Preceding period | Regression |
|-------------------|-------------------|----------------------------------|
| July 33 - June 40 | July 26 - June 33 | $\beta_2 = 0.320 + 0.714\beta_1$ |
| July 40 - June 47 | July 33 - June 40 | $\beta_2 = 0.265 + 0.750\beta_1$ |
| July 47 - June 54 | July 40 - June 47 | $\beta_2 = 0.526 + 0.489\beta_1$ |
| July 54 - June 61 | July 47 - June 54 | $\beta_2 = 0.343 + 0.677\beta_1$ |
| July 61 - June 68 | July 54 - June 61 | $\beta_2 = 0.399 + 0.546\beta_1$ |

So, if the observed value in 54 to 61 was 1.4, then our best estimate from 61 to 68 is $\beta_2 = 0.399 + 0.546 \times 1.4 = 1.1634$

Critique of Blume Adjustment

- Blume's method is pretty ad hoc. It does not have a firm mathematical foundation, but is just a generally plausible idea.
- It also does not take into account that for stocks with low idiosyncratic risk, the beta will be more accurate, which suggests less scaling for such stocks.
- Another issue with the Blume method is that it does not ensure that a portfolio with $\beta = 1$ and no idiosyncratic risk, i.e. the market!, is mapped to a portfolio with $\beta = 1$.

Fundamental analysis

- Another approach is to consider where does β come from?
- What makes one company riskier than another?
- One view is that it comes from the fundamental characteristics of the firm.
- If we define beta in terms of these, we get a measure that is much more reactive.

Example: If the company risks up its strategy by being much more leveraged, e.g. issuing a large amount of debt, it will immediately be reflected in the beta; while with the historical method it would take time to filter through to the measured values.

Vasicek Adjustment

- Vasicek has suggested a Bayesian technique that corrects this deficiency, as with all Bayesian techniques it requires the user to have a *prior* distribution, that is a view on what distribution of β s is reasonable.

$$\beta_{2i} = \frac{\sigma_{\beta_{1i}}^2}{\sigma_{\beta_{1i}}^2 + \sigma_{\bar{\beta}_1}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\beta_{1i}}^2 + \sigma_{\bar{\beta}_1}^2} \beta_{1i}$$

where $\bar{\beta}_1$ equal the average beta across a sample of stocks and $\sigma_{\bar{\beta}_1}^2$ is the variance of the distribution of historical estimates of beta over the same sample of stocks.

- Note $\sigma_{\bar{\beta}_1}^2$ is a measure of the uncertainty associated with the measurement of individual betas.
- For asset i , the Vasicek procedure involves taking a weighted average of $\bar{\beta}_1$ and the historical beta of asset i , β_{1i} .

Carrying out fundamental analysis

What sort of quantities could be regarded as important?

- Dividend to earnings ratio
- Asset growth, i.e., percentage increase
- Leverage, e.g. debt to equity ratio
- Earnings variability

How would we use fundamental characteristics in practice?

- Measure betas over some past period eg 5 years.
- Carry out linear regression of the betas against the fundamental characteristics.
- Use the regressed equation to predict the future beta for each firm.

Testing SFMs

- To test SFMs we can use the fact that, its assumptions imply

$$\rho_{ij} = \frac{\beta_i \beta_j \sigma_m^2}{\sigma_i \sigma_j}.$$

- Using this we can test betas through their performance as correlation coefficients' forecaster.
- Empirical evidence** based upon correlations shows that:
 - Non adjusted betas underestimate average correlations.
 - Blume's Technique tends to overestimate average correlations.
 - Vasicek's Technique leads to both underestimated and overestimated estimates, such that its final effect is not crystal clear.

SFM: ranking of risky assets

- Recall that in general, MVT does not allow for any ranking of risky assets.
- The good news is that – if we were willing to accept the SFM assumptions – ranking of risky assets is possible.

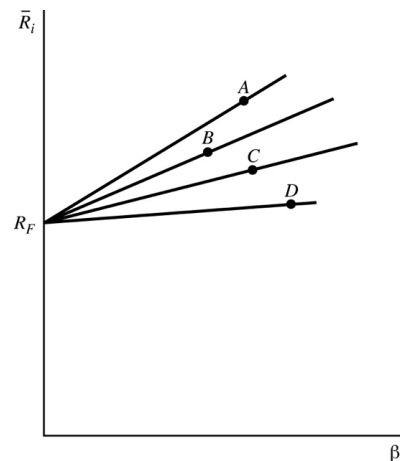
Theorem

Under the SFM assumptions, if a given risky asset belongs to the tangent portfolio, then all risky assets with higher Excess Return towards Beta (ERB) also belong to the tangent portfolio.

- This result considerably simplifies the calculations for finding **tangent portfolios**, even when shortselling is not allowed.
 - ⇒ It all reduces to a **three-step procedure**.

Finding Tangent Portfolios

Step 1: Ranking by $ERB_i = \frac{\bar{R}_i - R_f}{\beta_i}$



Example: step 1

Consider the following 10 risky assets ranked by excess return over beta (ERB), and suppose and $R_f = 5\%$ (both for lending and borrowing)

| Security No. i | Mean Return \bar{R}_i | Excess Return $\bar{R}_i - R_f$ | Beta β_i | Unsystematic Risk σ_{ei}^2 | Excess Return over Beta $\frac{\bar{R}_i - R_f}{\beta_i}$ |
|---------------------|----------------------------|------------------------------------|-------------------|--------------------------------------|--|
| 1 | 15 | 10 | 1 | 50 | 10 |
| 2 | 17 | 12 | 1.5 | 40 | 8 |
| 3 | 12 | 7 | 1 | 20 | 7 |
| 4 | 17 | 12 | 2 | 10 | 6 |
| 5 | 11 | 6 | 1 | 40 | 6 |
| 6 | 11 | 6 | 1.5 | 30 | 4 |
| 7 | 11 | 6 | 2 | 40 | 3 |
| 8 | 7 | 2 | 0.8 | 16 | 2.5 |
| 9 | 7 | 2 | 1 | 20 | 2 |
| 10 | 5.6 | 0.6 | 0.6 | 6 | 1.0 |

Finding Tangent Portfolios

Step 2: cut-off

- If shortselling is allowed, we know all assets belong to the tangent portfolio and

$$C^* = C_n = \frac{\sigma_m^2 \sum_{i=1}^n \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{ei}^2}}{1 + \sigma_m^2 \sum_{i=1}^n \frac{\beta_i^2}{\sigma_{ei}^2}}$$

- C^* is called the **cut-off level** and it tells us to take:
 - long positions in all assets with $SR_i > C^*$,
 - no investment if it happens $SR_i = C^*$, and
 - short positions in all assets with $SR_i < C^*$.

Finding Tangent Portfolios

Step 2: cut-off

- If shortselling is NOT allowed, we do not know how many assets belong to T . So, in Step 2, we need to proceed iteratively, starting from the asset with the highest ERB and moving downwards.
- We start by considering T has only one asset, $k = 1$, then $k = 2$, $k = 3$, etc.

- The cut-off C^* is defined

$$C^* = C_k = \frac{\sigma_m^2 \sum_{i=1}^k \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{ei}^2}}{1 + \sigma_m^2 \sum_{i=1}^k \frac{\beta_i^2}{\sigma_{ei}^2}} \quad \text{for } k \text{ s.t. } \begin{cases} ERB_i > C^* & i = 1, \dots, k \\ ERB_i < C^* & i = k + 1, \dots, n \end{cases}$$

and we stop at the first k that verifies the condition above.

- Only assets with $SR_i > C^*$ are included in the tangent portfolio T , so, k is its number of assets.

Example: step 2

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------------------|-------------------------------------|--|-----------------------------------|---|--|-------|
| Security No. i | $\frac{(\bar{R}_i - R_f)}{\beta_i}$ | $\frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{ei}^2}$ | $\frac{\beta_i^2}{\sigma_{ei}^2}$ | $\sum_{j=1}^i \frac{(\bar{R}_j - R_f)\beta_j}{\sigma_{ej}^2}$ | $\sum_{j=1}^i \frac{\beta_j^2}{\sigma_{ej}^2}$ | C_i |
| 1 | 10 | 2/10 | 2/100 | 2/10 | 2/100 | 1.67 |
| 2 | 8 | 4.5/10 | 5.625/100 | 6.5/10 | 7.625/100 | 3.69 |
| 3 | 7 | 3.5/10 | 5/100 | 10/10 | 12.625/100 | 4.42 |
| 4 | 6 | 24/10 | 40/100 | 34/10 | 52.625/100 | 5.43 |
| 5 | 6 | 1.5/10 | 2.5/100 | 35.5/10 | 55.125/100 | 5.45 |
| 6 | | 3/10 | 7.5/100 | 38.5/10 | 62.625/100 | 5.30 |
| 7 | | 3/10 | 10/100 | 41.5/10 | 72.625/100 | 5.02 |
| 8 | 2.5 | 1/10 | 4/100 | 42.5/10 | 76.625/100 | 4.91 |
| 9 | 2.0 | 1/10 | 5/100 | 43.5/10 | 81.625/100 | 4.75 |
| 10 | 1.0 | 0.6/10 | 6/100 | 44.1/10 | 87.625/100 | 4.52 |

$C^* = 5.45$ (shortselling not allowed) is indicated by a red arrow pointing to the value 5.45 in the C_i column.

$C^* = 4.52$ (shortselling allowed) is indicated by a red arrow pointing to the value 4.52 in the C_i column.

Finding Tangent Portfolios

Step 3: composition

- Given the cut-off level, we have the following formula for the entries of our usual vector Z

$$z_i = \frac{\beta_i}{\sigma_{ei}^2} \left(\frac{\bar{R}_i - R_f}{\beta_i} - C^* \right),$$

note C^* differs depending on whether we can or cannot shortsell.

- Recall $Z = \lambda X_T$, for λ constant and X_T the vector of weights T ,

- Shortselling allowed:

$$x_i^T = \frac{z_i}{\sum_{j=1}^n z_j}$$

- Shortselling allowed *a la* Lintner:

$$x_i^T = \frac{z_i}{\sum_{j=1}^n |z_j|}$$

- Shortselling not allowed:

$$x_i^T = \frac{z_i}{\sum_{j=1}^k z_j}$$

Example: step 3

| Security | Short Sales Disallowed | Lintner Definition of Short Sales | Standard Definition of Short Sales |
|----------|------------------------|-----------------------------------|------------------------------------|
| 1 | 23.5 | 8.0 | 647.1 |
| 2 | 24.6 | 9.5 | 770.6 |
| 3 | 20.0 | 9.0 | 729.4 |
| 4 | 28.4 | 21.5 | 1741.2 |
| 5 | 3.5 | 2.7 | 217.6 |
| 6 | 0 | -1.9 | -152.9 |
| 7 | 0 | -5.5 | -447.1 |
| 8 | 0 | -7.3 | -594.1 |
| 9 | 0 | -9.1 | -741.2 |
| 10 | 0 | -25.5 | -2070.6 |

Theory questions

- 1 What is a single factor model?
- 2 Derive expressions for expected return, variance and covariance.
- 3 Define specific risk, systematic risk and diversifiable risk.
- 4 Derive expressions for the beta and variance of a portfolio.
- 5 What happens to the variance of a large portfolio as the number of assets goes to infinity?

Theory questions

- 6 Explain how to find SFM parameters using linear regressions on historical market data.
- 7 How can we improve the stability of α and β estimates when using a single-factor model?
- 8 How could you assess if a single factor model is suitable for fitting a covariance matrix?
- 9 If we linearly regress the betas in one period against a previous period, what properties would we expect of the coefficients found?
- 10 Describe Blume's adjustment and discuss briefly why it is plausible.
- 11 Explain the idea underlying the Vasicek's adjustment.
- 12 How can a SFM be tested? Explain.

3.3 Multi-factor Models

- Learning objectives
- Multi-factor models
- Properties
- Model Types

Learning objectives

- motivate the use of multi-factor models,
- define a multi-factor model,
 - derive and use expressions for the expected returns, variances and covariances,
 - state how many parameters are required,
 - classify sorts of multi-factor models,
- discuss how to compare models for covariance matrices, and state some results,

The multi-factor model

- Let us consider a number of **uncorrelated** indices, I_j .

$$\text{Cov}(I_i, I_j) = 0 \quad \forall i, j$$

- We set

$$R_i = a_i + \sum_{k=1}^K b_{ik} I_k + c_i, \quad (1.1)$$

where the numbers a_i and b_{ij} are constants, while c_i are random but uncorrelated with the indices, I_k and with zero mean.

- As in the single-factor case, c_i expresses the idiosyncratic risk. We make the key simplifying assumption:

$$\mathbb{E}(c_i c_j) = 0.$$

- Without the **independence assumptions** – uncorrelated indices and specific components across assets – the model would not reduce complexity.

The need for multi-factor models

- A single-factor model may be too simplistic.
- Do we really believe that all correlation between stocks arises from the level of the market?
- Stocks in the same sector tend to have much more correlation than stocks from competing sectors.
- Stocks from different countries will have less correlation than stocks from the same country.
- One can attempt to introduce more complexity into the modelling by assuming return correlations can be explained by **more than one common factor**.

Model properties

Let us consider K indices, $k = 1, 2, \dots, K$, all uncorrelated with one another. We denote $\text{Var}(I_k) = \sigma_k^2$ and $\text{var}(c_i) = \sigma_{c_i}^2$.

In the context of MFM, we have:

$$\textcircled{1} \quad \mathbb{E}(R_i) = a_i + \sum_{k=1}^K b_{ik} \mathbb{E}(I_k),$$

$$\textcircled{2} \quad \text{Var}(R_i) = \sum_{k=1}^K b_{ik}^2 \sigma_k^2 + \sigma_{c_i}^2,$$

$$\textcircled{3} \quad \text{Cov}(R_i, R_j) = \sum_{k=1}^K b_{ik} b_{jk} \sigma_k^2$$

Q: How to get to the above expressions?

Expectation

- 1 The first of these follows from the linearity of expectation:

$$\begin{aligned}\mathbb{E}(R_i) &= \mathbb{E}(a_i) + \sum_{k=1}^K \mathbb{E}(b_{ik}I_k) + \mathbb{E}(c_i), \\ &= a_i + \sum_{k=1}^K b_{ik}\mathbb{E}(I_k).\end{aligned}$$

Covariance formula

- 3 For the covariance, we need to compute

$$\mathbb{E}((R_i - \mathbb{E}(R_i))(R_j - \mathbb{E}(R_j))).$$

This is equal to

$$\sigma_{ij} = \mathbb{E} \left(\left(c_i + \sum_{k=1}^K b_{ik}(I_k - \mathbb{E}(I_k)) \right) \left(c_j + \sum_{v=1}^K b_{jv}(I_v - \mathbb{E}(I_v)) \right) \right)$$

As with the variance, all cross terms disappear, and we obtain

$$\sigma_{ij} = \mathbb{E} \left(\sum_{k=1}^K b_{ik}(I_k - \mathbb{E}(I_k))b_{jk}(I_k - \mathbb{E}(I_k)) \right),$$

which is equal to

$$\sigma_{ij} = \sum_{k=1}^K b_{ik}b_{jk}\sigma_k^2.$$

Variance

- 2 For variance, we need to compute the expectation of $(R_i - \mathbb{E}(R_i))^2$. This is equal to

$$\sigma_i^2 = \mathbb{E} \left(\left(c_i + \sum_{k=1}^K b_{ik}(I_k - \mathbb{E}(I_k)) \right)^2 \right).$$

We can discard all cross terms because of zero correlation, so we get

$$\begin{aligned}\sigma_i^2 &= \mathbb{E}(c_i^2) + \sum_{k=1}^K b_{ik}^2 \mathbb{E}((I_k - \mathbb{E}(I_k))^2) \\ &= \sigma_{c_i}^2 + \sum_{k=1}^K b_{ik}^2 \sigma_k^2.\end{aligned}$$

Data requirements

We need less data than for a general model but more than for a single factor model. The model has

$$2n + 2K + Kn$$

parameters to calibrate.

OBS: If $n \sim 500$, and $K \sim 10$, this is a lot better than the general model but a lot worse than the single factor model.

Q: For a fixed number of risky assets n , what is the maximum number of factors K that makes sense to consider?

Types of multi-factor models

There are various different quantities, we could use as driving factors for our multi-factor model. These include

- Sector-based models.
- Macro-economic models.
- Statistical models.

One could also consider mixed models – models that are a mixture of the above types.

Macro-economic model

Assets' return correlations are driven by the wider economy.

Factors could include:

- price of oil,
- inflation,
- government bond yields,
- corporate bond yields (or spreads)
- economic growth.

Sector-based models

- Sector-based models are intuitive and popular.
- One has an index for the overall market and then one for each industrial sector.
- Examples of sectors are
 - banks,
 - oil,
 - pharmaceuticals,
 - steel.
- One could also extend this model to have factors for each country, as well for each sector.
 - ⇒ There is evidence that the market index accounts for about 30 – 50% of the variability of stocks, and that introducing industrial sectors explains another 10%.

OBS: The factors mentioned above do not fit our model definition, since they are not uncorrelated (but this can be tackled.)

Statistical factor models

- Rather than trying to find economically meaningful variables, one can instead apply statistical techniques to find the factors which explain most correlations.
- The standard technique is [principal components analysis](#). Essentially this means finding eigenvalues and eigenvectors of the covariance matrix, one keeps the two or three largest eigenvectors and then discards the rest.

Evaluating multi-factor models

- Are multi-factor models worth the effort? How can we answer this question?
- One way is to observe that they give a better fit to the historical correlation matrix.
 - that is not necessarily a good thing
 - if that were our sole criterion, we would just use the historical correlation matrix
- Remember a theory is assessed by taking the predictions it makes and experimentally testing them.
 - The quality of its assumptions is not necessarily what matters the most.
- A better criterion is: How well does the model predict future correlations?
 - This will determine how well the model does at suggesting good investments when using mean-variance analysis.

The need for orthogonal factors

- We assumed that the indices driving our multi-factor model were uncorrelated.
- This is unrealistic as for real-life choice of indices, these will always be correlated indices:

$$I_1^*, I_2^*, \dots, I_K^*$$

Examples:

- There is correlation between the US market index and the price of oil.
- There is correlation between the market and sector indices.
- etc.

The need for orthogonal factors

Theorem

For finite set of correlated indices, there is always an equivalent finite set of orthogonal factors.

- The standard procedure for removing correlation is the [Gram-Schmidt orthogonalization procedure](#).
- Gram-Schmidt is an algorithm designed for inner-product spaces.
- Gram-Schmidt procedure allows for quick orthogonalisation of factors.
- In many situations we can also use [linear regressions](#).

Example: two-factor models

Given a set of real-life (correlated indices) I_1^*, I_2^* , we can use linear regressions to get an equivalent set of uncorrelated indices I_1, I_2 :

$$R_i = a_i^* + b_{i1}^* I_1^* + b_{i2}^* I_2^* + c_i$$

Orthogonal Indexes

$$I_1 = I_1^*$$

$$I_2^* = \gamma_0 + \gamma_1 I_1 + d_i$$

$$d_i = I_2^* - (\gamma_0 + \gamma_1 I_1)$$

$$I_2 = d_i = I_2^* - (\gamma_0 + \gamma_1 I_1)$$

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + c_i$$

$$a_i = (a_i^* + b_{i2}^* \gamma_0)$$

$$b_{i1} = (b_{i1}^* + b_{i2}^* \gamma_1)$$

$$b_{i2} = b_{i2}^*$$

Theory questions

- 1 State the definition of multi-factor models for stock returns.
- 2 Derive the expected returns, variances and covariances of returns in multi-factor models for stock returns.
- 3 What are the advantages of multi-factor models over single-factor models for stock returns?
- 4 How many parameters are there in a multi-factor model?
- 5 Give three different ways to choose factors for a multi-factor model.