Risk Neutral modelling with exponential Lévy processes

João Guerra

CEMAPRE and ISEG, UTL

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Risk neutral valuation

- In an arbitrage-free market modeled by an exponential Lévy process (or exponential Lévy model), the price process of the underlying risky asset is given by
$$S_t = S_0 \exp(X_t),$$
where X_t is a Lévy process (i.e., essentially X_t has independent and stationary increments).
- In an exponential Lévy model, the discounted price process
$$\widetilde{S}_t = e^{-rt} S_t$$
is a martingale with respect to some martingale measure (or risk neutral measure) $\mathbb{Q}$.
- The value $\Pi_t(H_T)$ of a contingent claim (option of derivative) with payoff H_T, is given by the risk-neutral valuation formula:
$$\Pi_t(H_T) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[H_T | \mathcal{F}_t] \quad (1)$$

Risk neutral valuation in the Black-Scholes model

- Specifying an option pricing model is equivalent to specifying the law of S_t under the risk-neutral measure $\mathbb{Q}$.
- In the Black-Scholes model, the dynamics of S_t under $\mathbb{Q}$ can be defined by
$$dS_t = rS_t dt + \sigma S_t dW_t,$$
where W_t is a standard Brownian motion under $\mathbb{Q}$.
- Alternatively, we can define
$$S_t = S_0 \exp(X_t),$$
where $X_t = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t$.

Valuation with a risk neutral density

- For most exponential Lévy models, it is impossible to find a closed form solution, even for plain vanilla derivatives (the Black and Scholes model is an exception).
- We assume that the martingale measure $\mathbb{Q}$ has been chosen (the mean-correcting martingale measure, for example).
- Assume that we know the density $f_{\mathbb{Q}}$ of S_T under the equivalent risk neutral measure $\mathbb{Q}$. Then we have for the price of an European call with strike K and maturity T, at time 0 (see Eq. (1)):
$$\begin{aligned} C_0 &= \exp(-rT)\mathbb{E}_{\mathbb{Q}}\left[(S_T - K)^+\right] \\ &= \exp(-rT)\int_0^{+\infty} f_{\mathbb{Q}}(x)(x-K)^+ dx \\ &= \exp(-rT)\int_K^{+\infty} x f_{\mathbb{Q}}(x)\, dx - K\exp(-rT)\Pi_2, \end{aligned}$$
where Π_2 is the probability for the call option to be in the money at expiration.

Valuation with the Fourier transform

- For most of the Lévy distributions, this integral should be calculated numerically and this calculation can be computationally very demanding.
- Moreover, we may not know explicitly $f_{\mathbb{Q}} \Longrightarrow$ this method is of a limited interest in practice.
- The risk neutral density $f_{\mathbb{Q}}$ is rarely known, nevertheless we know from the Lévy-Khintchine formula the equation for the Fourier transform of S_t.
- In order to evaluate an option one then needs to invert the Fourier transform. The algorithms for the inversion of the Fourier transform are fast and optimized.
- The Fast Fourier transform (FFT) algorithm allows the calculation of the prices of options with different strikes in a single calculation.
- This method was developed by Carr and Madan in:
- Carr, P. et Madan D.B. Option valuation using the Fast Fourier Transform, Journal of Computational Finance, 2, pp 61-73.

Valuation with the Fourier transform

- Consider an European call with underlying S_t and with strike K. Define:

$$k = \ln(K),$$
$$s_T = \ln(S_T).$$

- Let $\Phi_T(u)$ be the characteristic function of s_T, i.e.

$$\Phi_T(u) = \mathbb{E}\left[e^{ius_T}\right] = \int_{-\infty}^{+\infty} e^{ius} q_T(s)\, ds, \tag{2}$$

where $q_T(s)$ is the density of s_T.

- The price of the call option at time 0 is:

$$\begin{aligned} C_0(k) &= \exp(-rT)\mathbb{E}_{\mathbb{Q}}\left[(S_T - K)_+\right] \\ &= \exp(-rT)\int_k^{\infty} \left(e^s - e^k\right) q_T(s)\, ds. \end{aligned} \tag{3}$$

Valuation with the Fourier transform

- The function $C_0(k)$ as a function of k is not square-integrable because as $k \to -\infty \Longrightarrow K \to 0$ and $C_0(k) \to S_0$ and therefore $C_0(k)$ is not integrable.
- But $C_0(k)$ as a function of k should be square-integrable in order to calculate the inverse Fourier transform.
- Carr and Madam suggested to consider a "modified call price" function:

$$c_0(k) = \exp(\alpha k) C_0(k),$$

with $\alpha > 0$ in order to ensure integrability when $k \to -\infty$.

Methods to valuate european derivatives in exponential Lévy models

Valuation with the Fourier transform

- The Fourier transform of $c_0(k)$ is

$$\Psi_T(v) = \int_{-\infty}^{+\infty} e^{ivk} c_0(k)\, dk \tag{4}$$

- Since $c_0(k) = \exp(\alpha k) C_0(k) \underset{k \to -\infty}{\approx} S_0 \exp(\alpha k)$, this function is square integrable in $-\infty$.

Valuation with the Fourier transform

- Inverting the Fourier transform, we obtain:

$$c_0(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivk}\Psi_T(v)\,dv,$$

$$C_0(k) = \frac{\exp(-\alpha k)}{2\pi}\int_{-\infty}^{+\infty} e^{-ivk}\Psi_T(v)\,dv.$$

- But $C_0(k)$ is real, and therefore:

$$\operatorname{Im}\left[\int_{-\infty}^{+\infty} e^{-ivk}\Psi_T(v)\,dv\right] = 0.$$

- Let $a(v)$ and $b(v)$ be the real and imaginary parts of $\Psi_T(v)$:

$$a(v) = \int_{-\infty}^{+\infty}\cos(vk)\,c_0(k)\,dk,$$

$$b(v) = \int_{-\infty}^{+\infty}\sin(vk)\,c_0(k)\,dk.$$

Valuation with the Fourier transform

- Then (note that a is even and b is odd)

$$\Psi_T(-v) = a(v) - ib(v).$$

Define the functions:

$$A(k) = \int_{-\infty}^{0} e^{-ivk}\Psi_T(v)\,dv$$

$$B(k) = 2\pi\exp(\alpha k)\,C_0(k) - A(k) = \int_{0}^{+\infty} e^{-ivk}\Psi_T(v)\,dv.$$

Valuation with the Fourier transform

- If we change the variable $v \to -v$ then

$$A(k) = \int_{+\infty}^{0} -e^{ivk}\Psi_T(-v)\,dv$$
$$= \int_{0}^{+\infty} \left[\cos(vk)\,a(v) + \sin(vk)b(v) + i\left(\sin(vk)a(v) - b(v)\cos(vk)\right)\right] dv.$$

On the other hand,

$$B(k) = \int_{0}^{+\infty} e^{-ivk}\Psi_T(v)\,dv$$
$$= \int_{0}^{+\infty} \left[\cos(vk)\,a(v) + \sin(vk)b(v) - i\left(\sin(vk)a(v) - b(v)\cos(vk)\right)\right] dv$$

- Comparing both expressions:

$$\operatorname{Re}[A(k)] = \operatorname{Re}[B(k)],$$
$$\operatorname{Im}[A(k)] = -\operatorname{Im}[B(k)]$$

Valuation with the Fourier transform

- Then, it is easy to see that

$$2\pi \exp(\alpha k)\, C_0(k) = A(k) + B(k)$$
$$= 2\operatorname{Re}[B(k)]$$

and therefore

$$C_0(k) = \frac{\exp(-\alpha k)}{\pi} \operatorname{Re}\left[\int_0^{+\infty} e^{-ivk}\Psi_T(v)\,dv\right]. \qquad (5)$$

- Now, let us try to express Ψ_T as a function of Φ_T. From (3) and (4), we have

$$\Psi_T(v) = e^{-rT}\int_{-\infty}^{+\infty}\int_{k}^{+\infty} e^{\alpha k} e^{ivk}\left(e^{s} - e^{k}\right) q_T(s)\,ds dk.$$

Valuation with the Fourier transform

- Using Fubini theorem and changing the order of integration, we have:

$$\begin{aligned}
\Psi_T(v) &= e^{-rT}\int_{-\infty}^{+\infty}\int_{-\infty}^{s}\left(e^{ivk+\alpha k+s}-e^{ivk+k(\alpha+1)}\right)q_T(s)\,dkds \\
&= e^{-rT}\int_{-\infty}^{+\infty} q_T(s)\left[\frac{e^{ivk+\alpha k+s}}{iv+\alpha}-\frac{e^{ivk+k(\alpha+1)}}{iv+\alpha+1}\right]_{-\infty}^{s} ds \\
&= e^{-rT}\int_{-\infty}^{+\infty} q_T(s)\left(\frac{e^{ivs+\alpha s+s}}{iv+\alpha}-\frac{e^{ivs+s(\alpha+1)}}{iv+\alpha+1}\right) ds \\
&= e^{-rT}\int_{-\infty}^{+\infty} q_T(s)\,e^{(iv+\alpha+1)s}\left(\frac{1}{(iv+\alpha)(iv+\alpha+1)}\right) ds \\
&= \frac{e^{-rT}}{\alpha^2+\alpha-v^2+iv(2\alpha+1)}\Phi_T(v-i(1+\alpha)), \qquad (6)
\end{aligned}$$

where Φ_T is the characteristic function of s_T - see (2).

Valuation with the Fourier transform

- We assume that $c_0(k)$ is integrable when $k\to+\infty$, i.e., we assume that $\Psi_T(0)=\int_{-\infty}^{+\infty} c_0(k)\,dk<\infty$. This condition in terms of Φ_T is

$$\Phi_T(-i(1+\alpha))<\infty$$

or $\int_{-\infty}^{+\infty} e^{(1+\alpha)s}q_T(s)\,ds<\infty$, which is equivalent to

$$\mathbb{E}\left[S_T^{1+\alpha}\right]<\infty.$$

- The final formula for the price of a call option in terms of Φ_T is (see (5) and (6))

$$C_0(k)=\frac{e^{-\alpha k}e^{-rT}}{\pi}\operatorname{Re}\left[\int_0^{+\infty}\frac{e^{-ivk}\Phi_T(v-i(1+\alpha))}{\alpha^2+\alpha-v^2+iv(2\alpha+1)}dv\right].$$

- Carr and Madan suggest to choose $\alpha\approx 0.25$. W. Schoutens proposes $\alpha\approx 0.75$. The choice of α affects the convergence speed.

FFT

- In order to calculate $C_0(k)$, we discretize the integral

$$C_0(k) = \frac{e^{-\alpha k} e^{-rT}}{\pi} \operatorname{Re}\left[\int_0^{+\infty} e^{-ivk} \Psi_T(v)\, dv\right] \approx \frac{e^{-\alpha k} e^{-rT}}{\pi} \operatorname{Re}\left[\int_0^{(N-1)\eta} e^{-ivk} \Psi_T(v)\, dv\right],$$

where η is the integration step and N is a large positive integer.

- Using the trapezoidal method for the integral approximation (with coefficients $\frac{1}{2}$ for the first and the last terms in the sum), we have

$$C_0(k) \approx \frac{e^{-\alpha k} e^{-rT}}{\pi} \operatorname{Re}\left[\sum_{j=0}^{N-1} e^{-iv_j k} \Psi_T(v_j) \cdot \eta \cdot w_j\right],$$

where $v_j = \eta \cdot j$,

$$w_j = \begin{cases} \frac{1}{2} & \text{if } j = 0 \text{ or } j = N-1 \\ 1 & \text{if } \quad 0 < j < N-1. \end{cases}$$

FFT

- We should center the analysis on the options around the options at-the money: $K = S_0$ or $k = \ln(S_0) := \theta$. Therefore, define

$$k_u = \theta - b + \lambda u, \quad u = 0, ..., N-1,$$
$$\lambda = \frac{2b}{N-1}.$$

- Therefore

$$C_0(k_u) \approx \frac{e^{-\alpha k} e^{-rT}}{\pi} \operatorname{Re}\left[\sum_{j=0}^{N-1} e^{-i\eta j(\theta - b + \lambda u)} \Psi_T(\eta j) \cdot \eta \cdot w_j\right] \approx \frac{e^{-\alpha k} e^{-rT}}{\pi} \eta \operatorname{Re}\left[\sum_{j=0}^{N-1} e^{-i\eta j \lambda u} \Psi_T(\eta j) \cdot e^{i\eta j(\theta - b)} \cdot w_j\right]$$

FFT

- With the Fast Fourier Transform algorithm (FFT), we can calculate the N values of the sum

$$w(u) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}ju} x(j), \quad u = 0, 1, ..., N-1,$$

with a number of product operations of $N\ln(N)$ instead of N^2.

- In order to apply the FFT algorithm, we must choose

$$\eta\lambda = \frac{2\pi}{N}.$$

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Schoutens, W. (2002). Lévy Processes in Finance: Pricing Financial Derivatives. Wiley, Section 2.5.