Models in Finance - Class 17

Master in Actuarial Science

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Black-Scholes model

- Assumptions underlying the Black-Scholes model:
  1. The price of the underlying share follows a geometric Brownian motion.
  2. There are no risk-free arbitrage opportunities.
  3. The risk-free rate of interest r is constant, the same for all maturities and the same for borrowing or lending (it is not critical and can be relaxed).
  4. Unlimited short selling (that is, negative holdings) is allowed.
  5. There are no taxes or transaction costs.
  6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

Black-Scholes model

- Key general implication of these assumptions: the market is complete: all derivatives have payoffs which can be replicated.
- Each of these assumptions is unrealistic to some degree:
- Share prices can jump $\Longrightarrow$ invalidates assumption 1 (gBm has continuous sample paths).
  Important consequence: it is not possible to rebalance the risk-free portfolio at every instant. The portfolio is not entirely risk-free. However, hedging strategies can still be constructed which substantially reduce the level of risk.
- The risk-free rate of interest does vary and in a unpredictable way. Over the short term of a typical derivative the assumption of a constant risk-free rate of interest is not far from reality. The model can be adapted in a simple way to allow for a stochastic risk-free rate. Different rates may apply for borrowing and lending.

Black-Scholes model

- Unlimited short selling may not be allowed except perhaps at penal rates of interest. These problems can be mitigated by holding mixtures of derivatives which reduce the need for short selling.
- Shares can normally only be dealt in integer multiples of one unit, not continuously and dealings attract transaction costs: invalidating assumptions 4, 5 and 6. Again we are still able to construct suitable hedging strategies which substantially reduce risk.
- Distributions of share returns tend to have fatter tails than suggested by the log-normal model, invalidating assumption 1.

Black-Scholes model

- Despite all the flaws in the model assumptions, analyses of market derivative prices indicate that the Black-Scholes can, in some situations, give a good approximation to the market.
- All models are only approximations to reality. It is always possible to take a model and show that its underlying assumptions do not hold in practice.
- A model is useful if, for a specified problem, it provides answers which are a good approximation to reality or if it provides insight into underlying processes.

BS model - underlying SDE

- Consider a European call option on a non-dividend paying share S_t driven by the SDE:

$$dS_t = \mu S_t dt + \sigma S_t dZ_t,$$

where Z_t is a standard Bm. μ and σ are referred as the drift and volatility parameters.
- Investors can also have holdings in a risk-free cash bond with price $B_t = B_0 e^{rt}$ where r is the assumed constant risk-free interest rate.
- The solution of the SDE is the gBm:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma Z_t\right).$$

- This price process is sometimes called a lognormal process, geometric Brownian motion or exponential Brownian motion.

BS model - BS formula

- Let $f(t,s)$ be the price at time t of a call option given:
  (i) the current share price $S_t = s$.
  (ii) the time of maturity $T > t$.
  (iii) the exercise price K.

Proposition (Black-Scholes formula):

$$f(t, S_t) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \tag{1}$$

where: $d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$ and $\Phi(z)$ is the cumulative distribution function of the standard normal distribution.

BS model - BS formula

- For a put option we also have the Black-Scholes formula:

$$f(t, S_t) = K e^{-r(T-t)} \Phi(-d_2) - S_t \Phi(-d_1). \tag{2}$$

- Exercise: Starting from the price of the call option given by the BS formula, use put-call parity to derive the formula for the price of the put option.
- We will discuss two proofs of this result for the call option, one using the partial differential equation (PDE) approach and the other using the martingale approach.