Models in Finance - Class 20

Master in Actuarial Science

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ISEG

Portfolio risk management

- Call options allow exposure to be gained to upside movements in the price of the underlying asset.
- Put options allow the downside risks to be removed.
- Combinations of various derivatives and the underlying asset into a portfolio allow us to modify our exposure to risk.
- In the proof of the Black-Scholes PDE we took a portfolio of a derivative and the underlying asset to create an instantaneously risk-free portfolio. This is called delta-hedging.
- Delta is one of what are called the Greeks.
- The Greeks are a group of mathematical derivatives which can be used to help us to manage or understand the risks in our portfolio.

The greeks

- Let $f(t,s)$ be the value at time t of a derivative when the price of the the underlying asset at t is $S_t = s$.
- Delta of the derivative:

$$\Delta = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial s}(t, S_t). \tag{1}$$

- The Δ of the underlying asset S_t is $\Delta = 1$ because $\frac{\partial S_t}{\partial S_t} = 1$.
- When we consider delta hedging we add up the deltas for the individual assets and derivatives (taking account, of course, of the number of units held of each). If this sum is zero and if the underlying asset prices follow a diffusion then the portfolio is instantaneously risk free.

The greeks

- If it is intended that the sum of the deltas should remain close to zero (this is what is called delta hedging) then normally it will be necessary to rebalance the portfolio. The extent of this rebalancing depends primarily on another greek: the Gamma Γ:

$$\Gamma = \frac{\partial^2 f}{\partial s^2} = \frac{\partial^2 f}{\partial s^2}(t, S_t) \tag{2}$$

- The Γ of the underlying asset S_t is $\Gamma = 0$ because $\frac{\partial^2 S_t}{\partial S_t^2} = 0$.
- Γ is the rate of change of Δ with the price of the underlying asset.

The greeks

- Suppose a portfolio is following a delta-hedging strategy. If the portfolio has a high value of Γ then it will require more frequent rebalancing or larger trades than one with a low value of Γ.
- Continuous rebalancing of the portfolio is not feasible and frequent rebalancing increases transaction costs.
- The need for rebalancing can be minimised by keeping $\Gamma \approx 0$.

The greeks

- Let us now define the vega (it is not the name of a greek letter):

$$\nu = \frac{\partial f}{\partial \sigma}. \tag{3}$$

- Vega is the rate of change of the price of the derivative with respect to a change in the volatility of S_t.
- For the underlying asset S_t the vega is zero.
- The value of a portfolio with a low value of vega will be relatively insensitive to changes in volatility.
- Therefore, it is less important to have an accurate estimate of σ if vega is low.
- Since σ is not directly observable, a low value of vega is important as a risk management tool.
- Furthermore, it is recognised that σ can vary over time. Since many derivative pricing models assume that σ is constant through time the resulting approximation will be better if vega is small.

The greeks

- Let us now introduce a new greek, the Rho ρ:

$$\rho = \frac{\partial f}{\partial r}. \tag{4}$$

- ρ measures the rate of change of the price of the derivative with respect to a change in the risk-free rate of interest.
- The risk-free rate of interest can vary by a small amount over the (usually) short term of a derivative contract. As a result a low value of ρ reduces risk relative to uncertainty in the risk-free rate of interest.

The greeks

- The greek Lambda λ:

$$\lambda = \frac{\partial f}{\partial q}, \tag{5}$$

where q is the assumed, continuous dividend yield on the underlying asset.
- λ measures the rate of change of the price of the derivative with respect to a change in the dividend yield on the underlying asset.

The greeks

- The greek Theta Θ:

$$\Theta = \frac{\partial f}{\partial t},$$

- Since time is a deterministic variable strictly increasing, it does not make sense to hedge against changes in t in the same way as we do for unexpected changes in the price of the underlying asset.
- The Black-Scholes PDE can be represented using the greeks:

$$\Theta + rs\Delta + \frac{1}{2}\sigma^2 s^2 \Gamma = rf. \qquad (6)$$

The greeks

- The Δ of a call option is positive and the Δ of a put option is negative.
- The Δ of the porfolio consisting of -1 derivative plus Δ shares is equal to:

$$(-1) \times \Delta + \Delta \times 1 = 0.$$

- Note that we can also construct a gamma-neutral portfolio, i.e., a portfolio with total gamma equal to zero.

The greeks

- If we consider the Black-Scholes formula and Garman-Kohlhagen formula, we can calculate the Δ by mathematical differentiation.
- We obtain, for a European Call:

$$\Delta = \Phi\left(d_1\right), \tag{7}$$

$$\Delta = e^{-q(T-t)}\Phi\left(d_1\right), \text{ in the dividend case.} \tag{8}$$

- For a European put:

$$\Delta = -\Phi\left(-d_1\right), \tag{9}$$

$$\Delta = -e^{-q(T-t)}\Phi\left(-d_1\right), \text{ in the dividend case.} \tag{10}$$

The greeks

- Exercise: Investor A has 10000 Euros invested in a portfolio of 1000 shares. Investor B has 10000 Euros invested in a portfolio of 5000 call options on the share and the delta is 0.5. If the share price increases by 10% what will be the value of each portfolio.
- Solution: Investor A portfolio value will be 11000 Eur (each share goes from 10 Eur to 11 Eur: increases 1 Eur).
  Investor B portfolio will be 12500 because each call option value will increase $1 \times \Delta = 0.5$ Eur, therefore each call increases from 2 Eur to 2.5 Eur.

The greeks

- Exercise: For each of the Greeks ν, Θ, ρ and λ discuss wether its value will be positive or negative in case of:
  (i) a call option.
  (ii) a put option.
- Exercise: Consider a call option and a put option on a dividend-paying security with the same maturity and exercise price. By considering the put-call parity for this case:
  (i) prove that $\Delta_c = \Delta_p + e^{-q(T-t)}$.
  (ii) prove that $\Gamma_c = \Gamma_p$.

The greeks

- Exercise: Examination Problem 1 (26 September 2007) (ii), (iii), (iv): work it out in detail.